Working Paper in Economics

202315

On Interim Rationalizable Monotonicity.

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On Interim Rationalizable Monotonicity*

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May 11, 2022

Abstract

Interim Rationalizable Monotonicity, due to Bergemann and Morris (2008a) and Oury and Tercieux (2012), fully characterizes the class of social choice functions that are implementable in interim rationalizable strategies by a mechanism that has a pure strategy Bayes-Nash equilibrium.

JEL classification: C79, D82

*Special thanks go to Yehuda (John) Levy, Siyang Xiong, and Anujit Chakraborty whose comments and suggestions have led to substantial improvements in the manuscript. We also thank Olivier Tercieux, Roberto Serrano, and audiences at the University of Glasgow, the National Taiwan University, the Economic Theory Workshop (University of York), and Conference on Economic Design. Ritesh Jain greatly acknowledges financial support from the Ministry of Science and Technology Grant, Taiwan under the grant: 110- 2410-H-001-082. The usual disclaimer applies.

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Electronic copy available at: https://ssrn.com/abstract=4106795

I. Introduction

A social choice function (SCF) f is (fully) interim rationalizably (and Bayes–Nash) implementable on a type space (T, κ) if there exists a mechanism such that (a) every interim rationalizable strategy profile leads to the realization of f and (b) it has a pure strategy Bayes–Nash equilibrium. We show that Interim Rationalizable Monotonicity (IRM, henceforth), due to Bergemann and Morris (2008a) and Oury and Tercieux (2012), fully characterizes the class of interim rationalizably (and Bayes–Nash) implementable functions on an arbitrary type space. IRM is a Bayesian version of the robust monotonicity condition introduced by Bergemann and Morris (2011) who study implementation in Belief-Free rationalizability.

Seminal contributions on interim rationalizable (and Bayes–Nash) implementation are Bergemann and Morris (2008a) and Oury and Tercieux (2012).[1] On a payoff type space, Bergemann and Morris (2008a) introduce IRM and show that IRM is necessary for interim rationalizable implementation by mechanisms satisfying the best reply property (henceforth, BRP.) [2] Oury and Tercieux (2012) extend IRM to an arbitrary type space and provide an intuitive discussion for the necessity of IRM.[3] Moreover, they show that IRM is also sufficient for interim rationalizable (and Bayes–Nash) implementation when combined with an auxiliary condition called Assumption 1. Oury and Tercieux (2012) and Bergemann and Morris (2008b) imply that, under Assumption 1, the requirement that the implementing mechanism has a pure strategy Bayes–Nash equilibrium is equivalent to restricting attention to an implementing

[1] The notion of interim rationalizable implementation has been introduced by (Bergemann and Morris (2008b)), whereas the notion of Bayes-Nash implementation can be found in (Jackson (1991)). By following Bergemann and Morris (2008a) and Oury and Tercieux (2012), we adopt the notion of implementation in interim rationalizable strategies and Bayes–Nash equilibrium strategies. In contrast to Bergemann and Morris (2008a) and Oury and Tercieux (2012), Kunimoto et al. (2020) study implementation problems in interim rationalizable strategies by dropping the existence requirement of a Bayes–Nash equilibrium. We discuss Kunimoto et al. (2020)'s contribution below.

[2] In this unpublished note, Bergemann and Morris (2008a) discuss how their work on robust implementation in general environments (Bergemann and Morris (2011)) can be adapted to the interim setup. We are grateful to Roberto Serrano for sharing this note with us.

[3] See footnote 4 (p. 1606) and the discussion on p. 1620. Though no formal arguments have been provided, the core arguments of the proof are contained in the proof of their Theorem 3 (Appendix B, p. 1629).

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mechanism satisfying the BRP (on this point, see Lemma 2 below).[4]

Beyond its relevance for implementation theory, our characterization result strengthens the connection between strict continuous implementation and interim rationalizable implementation. By adopting the notion of robustness of Weinstein and Yildiz (2007) in a mechanism design setting, Oury and Tercieux (2012) introduce the notion of strict continuous implementation and reach the following conclusion: When Assumption 1 holds, strict continuous implementation implies (full) implementation in interim rationalizable strategies. The sufficiency result of Oury and Tercieux (2012) discussed above is a key step in arriving at this conclusion. An SCF is strictly continuously implementable if there exists a strict Bayes–Nash equilibrium that continuously implements f.[5]

Specifically, Oury and Tercieux (2012) discuss that only functions satisfying IRM on (T, κ) are strictly continuously implementable on (T, κ). Moreover, they show that if f satisfies IRM on (T, κ) and it satisfies Assumption 1, then f is interim rationalizably implementable on (T, κ). Our characterization result strengthens Oury and Tercieux (2012)'s connection between partial implementation and full implementation as follows: Only interim rationalizably implementable functions on (T, κ) are strictly continuously implementable on (T, κ).

Roughly speaking, Assumption 1 is a condition that allows the planner to find a punishment outcome for each player, whatever the player's beliefs are. The assumption is satisfied in environments with transfers or bad outcomes that the planner does not desire. However, it may be violated in many environments, such as voting, matching, and allocation problems. For instance, Assumption 1 is violated when there is a state of the world at which a player deems all pure outcomes equally good. On this

[4] A sketch of the sufficiency proof is also provided by Bergemann and Morris (2008a) on a payoff type space. Proposition 5 of Bergemann and Morris (2008a) states that f is interim rationalizably implementable if it satisfies incentive compatibility, IRM, and an auxiliary condition called No Total Indifference (NTI, henceforth). Though the statement is not correct, Proposition 1 of Oury and Tercieux (2012) correct it by replacing NTI by its strengthening, represented by Assumption 1. Also see Section 8 of Kunimoto et al. (2020) for a thorough discussion.

[5] Specifically, Oury and Tercieux (2012) require that, in any type space that embeds (T, κ), there exists an equilibrium that (i) is a strict equilibrium on (T, κ), and (ii) it yields the desired outcome, not only at all types of (T, κ) but also at all types "close" to (T, κ).

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observation, in Section II, we present an interim rationalizably implementable voting rule violating Assumption 1. Moreover, it is violated in house allocation problems in which a player receives his worst house. This is the case in situations in which players have the same ranking of the houses.

As we discuss in Section II, Assumption 1 ensures that for every player, the elimination of a never-best reply starts in the first round of the iterative process of deletion of never-best replies. Indeed, the sufficiency result of Oury and Tercieux (2012) relies on this fact. However, Assumption 1 is not related to the assumption of common knowledge of rationality. Indeed, the iterative process that builds on the assumption of common knowledge of rationality neither requires deleting strategies simultaneously for all players nor requires deleting them in the first round for all players.

When Assumption 1 is relaxed, we show that IRM fully characterizes interim rationalizable implementation. This result is obtained by characterizing IRM in terms of an iterative condition, which embeds an argument of iterated deletion of never-best replies. This iterative condition is termed Interim Iterative Monotonicity (IIM, henceforth).

Recently, Xiong (2021) and Jain et al. (2022a) obtain full characterization results for rationalizable implementation of functions under complete information. The seminal paper on this class of implementation problems is Bergemann et al. (2011), which critically hinges on a condition similar to Assumption 1, named No Worst Alternatives (NWA, henceforth) and on the assumption that there are three or more players.

Xiong (2021) and Jain et al. (2022a) show that the idea of using iterative arguments is fruitful in both dispensing with the NWA condition and relaxing the assumption of three or more players. Indeed, Jain et al. (2022a) offer a novel iterative condition, named Iterative Monotonicity, and they provide an iterative characterization of rationalizably implementable functions under complete information when there are two or more players.[6] IIM is the counterpart of iterative monotonicity in an incomplete information setup.

Following Jain et al. (2022a), IIM is defined on the space of deception profiles, over

[6]Xiong (2021) provides a complete characterization of rationalizably implementable functions when there are three or more players.

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which we define a decreasing sequence of deception profiles $\left(\beta^k\right)_{k\geq 0}$ (in the sense of set inclusion). The limit of the sequence, which we refer to as β^*, can be viewed as the profile of the largest deceptions that the planner cannot rule out in *any* implementing mechanism. An SCF f satisfies IIM on a type space (T, κ) if for any type profiles t and t' such that $t' \in \beta^*(t)$, it holds that $f(t) = f(t')$. IIM is a measurability condition, which is reminiscent of the classical Abreu–Matsushima measurability condition (Abreu and Matsushima (1992)).[7]

As is typical in the implementation literature, the sufficiency result of Oury and Tercieux (2012) is based on designing an "augmented" direct mechanism. However, the devised mechanism does not work when Assumption 1 is relaxed. Indeed, thanks to the assumption, the augmentation of the direct mechanism used by Oury and Tercieux (2012) relies on β^0, which is the first element of the sequence $\left(\beta^k\right)_{k\geq 0}$. However, our characterization result is obtained by devising an augmentation of the direct mechanism that may crucially hinge on the entire sequence. Therefore, we provide an iterative characterization of interim rationalizably implementable functions.

Our result shows that IRM is necessary and sufficient for interim rationalizable implementation by a mechanism that has a pure strategy Bayes-Nash equilibrium. However, Kunimoto et al. (2020) show that IRM is not a necessary condition for implementation in interim rationalizable strategies when the existence requirement of a Bayes–Nash equilibrium (pure and mixed) is dropped. Indeed, Kunimoto et al. (2020) show that a weakening of IRM, called weak IRM (w-IRM, henceforth), is necessary for implementation. Moreover, they show that w-IRM, when combined with a weakening of Assumption 1, termed weak no-worst-rule (w-NWR, henceforth), is also sufficient. In Appendix B, we show that w-IRM fully identifies the class of implementable functions in interim rationalizable strategies. This result is achieved by showing that w-IRM is equivalent to a weakening of IIM, which, in turn, is shown to be sufficient. This characterization result shows that the extra constraints imposed by IRM (relative to w-IRM) are due to the constraints imposed by the existence

[7]Abreu and Matsushima (1992) proposed a measurability condition, now referred to as Abreu–Matsushima measurability, to characterize virtual rationalizable implementation when there is incomplete information.

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requirement of Bayes–Nash equilibria.[8]

Section II present our motivating example. Section III presents the implementation model. Section IV discusses IIM and relates it to IRM. Section V presents our characterization result. Section VI concludes. Appendices include proofs not in the main body.

II. Motivating example

Suppose that there are two players, player 1 and player 2. Assume that the sets of types are $\Theta_1 = \{\theta_1, \theta_1'\}$ for player 1 and $\Theta_2 = \{\theta_2, \theta_2'\}$ for player 2. The possible type profiles in $\Theta_1 \times \Theta_2$ are (θ_1, θ_2), (θ_1', θ_2), (θ_1, θ_2') and (θ_1', θ_2'). Let $\phi \in \Delta(\Theta_1 \times \Theta_2)$ be the common prior and assume that the type profiles (or states) are equally likely, that is, $\phi(\theta) = \frac{1}{4}$ for all $\theta \in \Theta_1 \times \Theta_2$. The type $\hat{\theta}_i \in \Theta_i$ is only observed by player i, who uses this information both to make decisions and to update his beliefs about the likelihood of his opponent's types (using the conditional probability $\phi\left(\hat{\theta}_j | \hat{\theta}_i\right)$). The set of pure outcomes is given by $A = \{a, b, c, d\}$. For player $i = 1, 2$, let $\upsilon_i : \Delta(A) \times \Theta_1 \times \Theta_2 \to \mathbb{R}$ be the state-dependent payoff function of player i. For each $\theta \in \Theta_1 \times \Theta_2$, $\upsilon_i(\cdot, \theta)$ satisfies the expected utility hypothesis for $i = 1, 2$. Players' state-dependent payoff functions over A are represented in the table below.

(θ_1, θ_2)		(θ_1', θ_2)		(θ_1, θ_2')		(θ_1', θ_2')	
υ_1	υ_2	υ_1	υ_2	υ_1	υ_2	υ_1	υ_2
a, b, c, d	a	a, b, c, d	a	c	c	d	c
	c		c	d	a	c	a
	b		b	a, b	d	a, b	d
	d		d		b		b

where, as usual, $\substack{\alpha \\ \beta}$ for player i in state θ means that he strictly prefers α to β in state θ, while α, β in state θ means that this i is indifferent between α and β in state θ.

Suppose that we want to implement f in interim correlated rationalizable strategies,

[8] w-IRM is a Bayesian version of the weak robust monotonicity condition introduced by Kunimoto and Saran (2020).

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where $f(\theta_1, \theta_2) = a$, $f(\theta_1', \theta_2) = b$, $f(\theta_1, \theta_2') = c$ and $f(\theta_1', \theta_2') = d$. To this end, let us consider the following direct mechanism, where player 1 is the row player and player 2 is column player.

	θ_2	θ_2'
θ_1	a	c
θ_1'	b	d

To show that the direct mechanism implements f, let us note that truth-telling is always the unique dominant strategy for player 2. Consequently, truth-telling is the only interim correlated rationalizable strategy for player 1. Observe that truth-telling is also the Bayes—Nash equilibrium of game. Thus, the above mechanism implements f in interim correlated rationalizable strategies and Bayes—Nash equilibrium strategies.

However, in this example, Assumption 1 of Oury and Tercieux (2012) is violated. This assumption is formally stated in Definition 6. The easiest way to see it is to recall that this assumption implies the condition of no total indifference. In our example, this condition requires that no player is indifferent over the entire set A at any state: for all $i = 1, 2$ and all $\theta \in \Theta_1 \times \Theta_2$, there exist $x, y \in A$ such that such that $\upsilon_i(x, \theta) \neq \upsilon_i(x, \theta)$. As it can be checked from the above table, player 1 is indifferent over the entire set A at states (θ_1, θ_2) and (θ_1', θ_2).

III. The Implementation Model

Preliminaries

Throughout the paper, if X is a topological space, we treat it as a measurable space with its Borel sigma field, and the space of Borel probability measures on X is denoted by $\Delta(X)$. Spaces $\Delta(X)$ are endowed with the topology of weak convergence of measures. Throughout the paper, we treat each countable set as a topological space endowed with the discrete topology. A subset Y of a topological space X is a dense subset of X if the closure of Y in X is equal to X. With abuse of notation, given a space X, let δ_x denote a degenerate distribution in $\Delta(X)$ assigning probability 1 to

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$\{x\}$.

We consider a finite set of players $\mathcal{I} = \{1, ..., I\}$. Each player i has a bounded utility function $u_i : \Delta(A) \times \Theta \to \mathbb{R}$ where A is the set of (pure) outcomes and Θ is the set of states (of nature). For each $\theta \in \Theta$, $u_i(\cdot, \theta)$ satisfies the expected utility hypothesis. We assume that Θ and A are countable and hence are separable metric spaces.

Throughout the paper, if, for each $i \in \mathcal{I}$, there is a space X_i, we write X as an abbreviation for $\Pi_{i \in \mathcal{I}} X_i$ and, for each $i \in \mathcal{I}$, X_{-i} for $\Pi_{j \in \mathcal{I} \setminus \{i\}} X_j$.

A *model* (of incomplete information) is a pair $\mathcal{T} \equiv (T, \kappa)$, where $T = \Pi_{i \in \mathcal{I}} T_i$ is a countable type space and, for each $i \in \mathcal{I}$, $\kappa(t_i) \in \Delta(\Theta \times T_{-i})$ denotes the associated beliefs for each type $t_i \in T_i$ of player i satisfying the following condition: For all $t_i \in T_i$, $\text{Supp}(\kappa(t_i)) = \Delta(\Theta \times T_{-i})$.

A typical type profile of T (*resp.*, T_{-i}) is denoted by t (*resp.*, t_{-i}). Throughout the paper, we rule out the case that $\mathcal{T}$ is a model of complete information, for the sake of simplicity.

A (stochastic) *mechanism* is a pair $\mathcal{M} \equiv (M, g)$, where $M \equiv \prod_{i \in \mathcal{I}} M_i$ is a message space and the outcome function $g : M \to \Delta(A)$ assigns to every $m \in M$ an element of $\Delta(A)$. For each $i \in \mathcal{I}$, M_i is player i's message space, which is assumed to be a (nonempty) countable set. A message profile $m \in M$ is often written as (m_i, m_{-i}), where $m_{-i} \in M_{-i}$.

Müller (2020) shows that the restricting attention to countable mechanisms for robust implementation is without loss of generality. Kunimoto et al. (2020) provide the same result for the interim setup.[9] For a further discussion we refer the reader to Remark 3 below.

Solution concepts

Given a mechanism $\mathcal{M}$ and a model $\mathcal{T}$, $U(\mathcal{M}, \mathcal{T})$ denotes the induced game of incomplete information. In this game, a (behavioral) strategy of player i is any function $\sigma_i : T_i \to \Delta(M_i)$. We write $\sigma_i(t_i)[m_i]$ for the probability that σ_i assigns to m_i

[9]See Theorem 8.1, p. 45, of Kunimoto et al. (2020).

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when player i is of type t_i. Player i's strategy σ_i is a pure strategy if $\sigma_i : T_i \to M_i$. Given a mechanism $\mathcal{M}$, for each $i \in \mathcal{I}$, player i's best reply correspondence BR_i from $\Delta(\Theta \times M_{-i})$ to M_i be defined, for all $\pi_i \in \Delta(\Theta \times M_{-i})$, by

$$BR_i(\pi_i|\mathcal{M}) = \arg\max_{m_i \in M_i} \left(\sum_{(\theta, m_{-i}) \in \Theta \times M_{-i}} \pi_i[\theta, m_{-i}][u_i(g(m_i, m_{-i}), \theta)] \right).$$

Since we allow for infinite mechanisms, the correspondence may be empty. For all $i \in I$, all $t_i \in T_i$ and all $\sigma_{-i} \equiv (\sigma_j)_{j \in \mathcal{I} \setminus \{i\}}$, we write $\pi_i(t_i, \sigma_{-i}) \in \Delta(\Theta \times M_{-i})$ for the joint distribution on the underlying uncertainty and the messages of other players induced by t_i and σ_{-i}.[10]

Definition 1. Let $U(\mathcal{M}, \mathcal{T})$ be any game of incomplete information. A profile of strategies $\sigma = (\sigma_i)_{i \in \mathcal{I}}$ is a *Bayes—Nash equilibrium* of $U(\mathcal{M}, \mathcal{T})$ if, for all $i \in \mathcal{I}$ and all $t_i \in T_i$,

$$m_i \in \text{Supp}(\sigma_i(t_i)) \implies m_i \in BR_i(\pi_i(t_i, \sigma_{-i})|\mathcal{M}).$$

We denote by $BNE(U(\mathcal{M}, \mathcal{T}))$ the set of all pure strategy Bayes—Nash equilibria of $U(\mathcal{M}, \mathcal{T})$. To distinguish between pure strategy and mixed strategy equilibrium, let us denote by $\overline{BNE}(U(\mathcal{M}, \mathcal{T}))$ as the set of pure strategy Bayes—Nash equilibria of $U(\mathcal{M}, \mathcal{T})$.

Next, let us define the solution concept of interim correlated rationalizability (ICR, henceforth), which was introduced by Dekel et al. (2007). Before introducing it, we need additional notation. Fix any pair $(\mathcal{M}, \mathcal{T})$. For all $i \in \mathcal{I}$, let Σ_i be a nonempty correspondence from T_i to $2^{M_i} \setminus \{\emptyset\}$, and let $\mathfrak{S}_i^{\mathcal{M}, \mathcal{T}}$ denote the set of all nonempty correspondences from T_i to $2^{M_i} \setminus \{\emptyset\}$. Let $\mathfrak{S}^{\mathcal{M}, \mathcal{T}} = \Pi_{i \in \mathcal{I}} \mathfrak{S}_i^{\mathcal{M}, \mathcal{T}}$, with Σ as a typical profile of $\mathfrak{S}^{\mathcal{M}, \mathcal{T}}$. For all $i \in \mathcal{I}$ and all $t_i \in T_i$, let $\Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i})$ be defined by

$$\Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i}) = \left\{ \pi_i \in \Delta(\Theta \times T_{-i} \times M_{-i}) \,|\, \text{marg}_{\Theta \times T_{-i}} \pi_i = \kappa(t_i) \right\},$$

[10]Formally, $\pi_i(t_i, \sigma_{-i}) \in \Delta(\Theta \times M_{-i})$ is defined by $\pi_i(t_i, \sigma_{-i}) = \sum_{t_{-i} \in T_{-i}} \kappa(t_i)[\theta, t_{-i}] \sigma_{-i}(t_{-i})[m_{-i}]$, where $\kappa(t_i)[\theta, t_{-i}]$ is the probability attached to $[\theta, t_{-i}]$ under $\kappa(t_i)$, and $\sigma_{-i}(t_{-i})[m_{-i}]$ is the probability attached to m_{-i} under $\sigma_{-i}(t_{-i})$.

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and, for all $\Sigma_{-i} \in \mathfrak{S}_{-i}^{\mathcal{M},\mathcal{T}}$, let $\Delta^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i})$ be defined by

$$\Delta^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i}) = \left\{ \pi_i \;\middle|\; \begin{array}{l} \pi_i \in \Delta(\Theta \times T_{-i} \times M_{-i}) \text{ and} \\ \pi_i[\theta, t_{-i}, m_{-i}] > 0 \implies m_{-i} \in \Sigma_{-i}(t_{-i}) \end{array} \right\}.$$

For the sake of brevity, we write $\Delta_{t_i}^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i})$ for $\Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i}) \cap \Delta^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i})$.

For all $(\mathcal{M}, \mathcal{T})$ and all $\Sigma \in \mathfrak{S}^{\mathcal{M},\mathcal{T}}$, Σ is a *best-reply set* in $U(\mathcal{M}, \mathcal{T})$ if, for all $i \in \mathcal{I}$, all $t_i \in T_i$ and all $m_i \in \Sigma_i(t_i)$, there exists

$$\pi_i(t_i) \in \Delta_{t_i}^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i})$$

such that

$$m_i \in BR_i\left(\text{marg}_{\Theta \times M_{-i}} \pi_i(t_i) | \mathcal{M}\right).$$

Definition 2. For all $(\mathcal{M}, \mathcal{T})$, all $i \in \mathcal{I}$ and all $t_i \in T_i$, the *set of interim correlated rationalizable messages at type* t_i, denoted by $S_i^{\mathcal{M},\mathcal{T}}(t_i)$, is defined by

$$S_i^{\mathcal{M},\mathcal{T}}(t_i) = \{m_i \in \Sigma_i(t_i) \,|\text{for some best-reply set } \Sigma \text{ in } U(\mathcal{M}, \mathcal{T})\}.$$

For all $t \in T$, we write $S^{\mathcal{M},\mathcal{T}}(t)$ for $\Pi_{i \in \mathcal{I}} S_i^{\mathcal{M},\mathcal{T}}(t_i)$.

Alternatively, the set of interim correlated rationalizable messages can be computed iteratively, where transfinite induction may be necessary to reach the fixed point. Following Aliprantis and Border (2006), we denote by Ω the set whose elements are called ordinals, which are ordered by $\leq$. The set Ω is such that (i) it is uncountable and (ii) it has a greatest element ω_1.[11]

Definition 3. For all $(\mathcal{M}, \mathcal{T})$, all $i \in \mathcal{I}$ and all $t_i \in T_i$, let $S_i^{0,\mathcal{M},\mathcal{T}}(t_i) = M_i$ and, for all ordinal numbers $\alpha \in \Omega \backslash \{0\}$, define $S_i^{\alpha,\mathcal{M},\mathcal{T}}(t_i)$ as follows:

[11]The existence of this set Ω is proved in Theorem 1.14 of Aliprantis and Border (2006) p. 19.

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- If α is a successor ordinal, then

$$S_i^{\alpha,\mathcal{M},\mathcal{T}}(t_i) = \left\{ m_i \in S_i^{\alpha-1,\mathcal{M},\mathcal{T}}(t_i) \;\middle|\; \begin{array}{l} \text{There exists } \pi_i(t_i) \in \Delta_{t_i}^{S_{-i}^{\alpha-1,\mathcal{M},\mathcal{T}}}(\Theta \times T_{-i} \times M_{-i}) \\ \text{such that } m_i \in BR_i\left(\text{marg}_{\Theta\times M_{-i}}\pi_i(t_i)|\mathcal{M}\right). \end{array} \right\}$$

- If α is a limit ordinal, then

$$S_i^{\alpha,\mathcal{M},\mathcal{T}}(t_i) = \bigcap_{\gamma<\alpha} S_i^{\gamma,\mathcal{M},\mathcal{T}}(t_i),$$

Let $S_i^{\infty,\mathcal{M},\mathcal{T}}(t_i) = \bigcap_{\alpha\in\Omega} S_i^{\alpha,\mathcal{M},\mathcal{T}}(t_i)$ be the set of interim correlated rationalizable messages at type t_i.

Arieli (2010) shows that the correspondence $S^{\infty,\mathcal{M},\mathcal{T}} = \Pi_{i\in\mathcal{I}} S_i^{\infty,\mathcal{M},\mathcal{T}}$ is a best-reply set of $U(\mathcal{M},\mathcal{T})$, that is, for all $i \in \mathcal{I}$, $S_i^{\infty,\mathcal{M},\mathcal{T}} \subseteq S_i^{\mathcal{M},\mathcal{T}}$. Indeed, Arieli (2010) shows the following result.

Lemma 1 (Arieli (2010), Theorem 1, p. 914)**.** For all $(\mathcal{M},\mathcal{T})$, all $i \in \mathcal{I}$ and all $t_i \in T_i$, there exists a least ordinal number α such that

$$S_i^{\alpha,\mathcal{M},\mathcal{T}}(t_i) = S_i^{\alpha+1,\mathcal{M},\mathcal{T}}(t_i) = S_i^{\mathcal{M},\mathcal{T}}(t_i). \tag{1}$$

Implementation

Let $\mathcal{T}$ be given. A (stochastic) *social choice function* (SCF, henceforth) is a function $f : T \to \Delta(A)$. Following Oury and Tercieux (2012), we assume that the planner cares about all profiles of types in T.

Definition 4. A mechanism $\mathcal{M}$ *implements* $f : T \to \Delta(A)$ *in interim correlated rationalizable strategies* (ICR-implements, henceforth) on $\mathcal{T}$ if the following two conditions are satisfied.

(i) For all $i \in \mathcal{I}$ and all $t_i \in T_i$, $S_i^{\mathcal{M},\mathcal{T}}(t_i) \neq \emptyset$.

(ii) For all $t \in T$, $m \in S^{\mathcal{M},\mathcal{T}}(t) \implies g(m) = f(t)$.

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If such a mechanism exists, f is *interim correlated rationalizably* (ICR, henceforth) *implementable*, or simply, *ICR-implementable* on $\mathcal{T}$.

In a complete information environment, Xiong (2021) and Jain et al. (2022a) fully characterize the class of implementable functions in rationalizable strategies. Their results show that every implementable function in rationalizable strategies is also Nash implementable. The reason is that the implementing mechanism in rationalizable strategies never fails to have a Nash equilibrium. This is not the case in incomplete information environments, in which implementing mechanisms may fail to have Bayes—Nash equilibria.[12] Following Oury and Tercieux (2012), we assume that the planner is interested in implementing in interim correlated rationalizable strategies and Bayes–Nash equilibria.

Definition 5. A mechanism $\mathcal{M}$ *implements* $f : T \to \Delta(A)$ on $\mathcal{T}$ *in Bayes–Nash equilibria* if (i) $\overline{BNE}(U(\mathcal{M},\mathcal{T})) \neq \emptyset$ and (ii) for all $\sigma \in BNE(U(\mathcal{M},\mathcal{T}))$ and for all $t \in T$, $\bigcup_{m \in Supp(\sigma(t))} g(m) = f(t)$.

Remark 1. It is clear from the definition of Bayes–Nash equilibrium that for any $\sigma \in BNE(U(\mathcal{M},\mathcal{T}))$ and for any $t \in T$, $\text{Supp}(\sigma(t)) \subseteq S^{\mathcal{M},\mathcal{T}}(t)$.

Thus, Definition 4 implies part (ii) of the definition above. Thus, a mechanism $\mathcal{M}$ that implements an f in interim rationalizable strategies also implements f in Bayes–Nash equilibrium if and only if $\overline{BNE}(U(\mathcal{M},\mathcal{T})) \neq \emptyset$.

The lemma below shows that a mechanism $\mathcal{M}$ that ICR-implements f also implements f in Bayes–Nash equilibrium if and only if $\mathcal{M}$ satisfies the *Equilibrium Best-Response Property* (EBRP). A mechanism $\mathcal{M}$ satisfies the EBRP on $\mathcal{T}$ if there exists a pure strategy profile σ such that for all $t \in T$,

$$\sigma(t) \in S^{\mathcal{M},\mathcal{T}}(t),$$

[12]By assuming a variant of Assumption 1 of Oury and Tercieux (2012), Kunimoto et al. (2020) study implementation problems in interim rationalizable strategies without requiring the existence of Bayes–Nash equilibria. See Section VI for a further discussion.

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and for all $i \in \mathcal{I}$ and all $t_i \in T_i$,

$$BR_i\left(\pi_i\left(t_i, \sigma_{-i}\right) | \mathcal{M}\right) \neq \emptyset.$$

The EBRP is a variant of the s-interim best reply property introduced in Jain et al. (2022b) to fully characterize robust implementation in terms of (robust) rationalizable implementation. EBRP is a weakening of the best reply property (henceforth BRP) introduced by Bergemann and Morris (2008b) (See Definition 4, p. 6).[13]

Lemma 2. Assume that $\mathcal{M}$ ICR-implements f on $\mathcal{T}$. $\mathcal{M}$ implements f on $\mathcal{T}$ in Bayes–Nash equilibria if and only if $\mathcal{M}$ satisfies the EBRP.

Proof. Assume that $\mathcal{M}$ ICR-implements f on $\mathcal{T}$. Assume that $\mathcal{M}$ satisfies the EBRP on $\mathcal{T}$. Let us show that $\mathcal{M}$ implements f on $\mathcal{T}$ in Bayes–Nash equilibria. To this end, we need only to show that $\overline{BNE}\left(U\left(\mathcal{M}, \mathcal{T}\right)\right) \neq \emptyset$. Since $\mathcal{M}$ ICR-implements f and $\mathcal{M}$ satisfies the EBRP, it follows that there exists a pure strategy profile σ such that for all $t \in T$, $\sigma\left(t\right) \in S^{\mathcal{M},\mathcal{T}}\left(t\right)$, and for all $i \in \mathcal{I}$ and all $t_i \in T_i$, $BR_i\left(\pi_i\left(t_i, \sigma_{-i}\right) | \mathcal{M}\right) \neq \emptyset$. Let us show that $\sigma \in \overline{BNE}\left(U\left(\mathcal{M}, \mathcal{T}\right)\right)$.

For all $i \in \mathcal{I}$ and all $t_i \in T_i$, since $BR_i\left(\pi_i\left(t_i, \sigma_{-i}\right) | \mathcal{M}\right) \neq \emptyset$, let $\hat{\sigma}_i\left(t_i\right) \in BR_i\left(\pi_i\left(t_i, \sigma_{-i}\right) | \mathcal{M}\right)$ for all $t_i \in T_i$ and all $i \in \mathcal{I}$. Fix any $i \in \mathcal{I}$. By construction, we see that for all $t \in T$, $\left(\hat{\sigma}_i\left(t_i\right), \sigma_{-i}\left(t_{-i}\right)\right) \in S^{\mathcal{M},\mathcal{T}}\left(t\right)$. Moreover, since $\mathcal{M}$ ICR-implements f on $\mathcal{T}$, we also have that for all $t \in T$, $f\left(t\right) = g\left(\hat{\sigma}_i\left(t_i\right), \sigma_{-i}\left(t_{-i}\right)\right)$. Thus, we can replace $\hat{\sigma}_i$ with σ_i and see that $\sigma_i\left(t_i\right) \in BR_i\left(\pi_i\left(t_i, \sigma_{-i}\right) | \mathcal{M}\right)$ for all $t_i \in T_i$. Since the choice of i was arbitrary, we have that $\sigma \in \overline{BNE}\left(U\left(\mathcal{M}, \mathcal{T}\right)\right)$.

For the converse, assume that $\mathcal{M}$ implements f on $\mathcal{T}$ in Bayes–Nash equilibria. This implies that $\overline{BNE}\left(U\left(\mathcal{M}, \mathcal{T}\right)\right) \neq \emptyset$. Thus, $\mathcal{M}$ satisfies the EBRP on $\mathcal{T}$. □

IV. Interim Iterative Monotonicity

In the following section, we present our necessary condition. Let $\mathcal{T}$ be any model. For every player $i \in \mathcal{I}$, let us call any map $\beta_i : T_i \to 2^{T_i} \setminus \{\emptyset\}$ as player i's deception. A

[13] Variants of EBRP have been used in the literature to avoid technical issues, such as Bergemann et al. (2011), Bergemann and Morris (2011).

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special deception for player i is the truth-telling deception, β_i^t, defined by $\beta_i^t(t_i) = \{t_i\}$ for all $t_i \in T_i$. Another special deception for player i is denoted by $\bar{\beta}_i$ and defined by $\bar{\beta}_i(t_i) = T_i$. For any β_i and β_i' we write $\beta_i \subseteq \beta_i'$ if $\beta_i(t_i) \subseteq \beta_i'(t_i)$ for all $t_i \in T_i$. Let $\mathcal{B}_i$ be the set of all player i's deceptions containing the truth-telling deception; that is,

$$\mathcal{B}_i = \left\{\beta_i : T_i \to 2^{T_i} \setminus \{\emptyset\} \,|\beta_i^t \subseteq \beta_i\right\}.$$

Let $\mathcal{B} = \Pi_{i\in\mathcal{I}}\mathcal{B}_i$, with $\beta = (\beta_i)_{i\in\mathcal{I}}$ as a typical deception profile of $\mathcal{B}$.

For every $i \in \mathcal{I}$, let Y_i^f be the set of mappings from T_{-i} to $\Delta(A)$ satisfying the following requirement. Whatever is player i's actual type, he would never prefer the outcome to be selected by a mapping Y_i^f to the outcome he could obtain under f if all his opponents were reporting truthfully. Formally,

$$Y_i^f = \left\{ y : T_{-i} \to \Delta(A) \;\middle|\; \begin{array}{l} \text{For all } \tilde{t}_i \in T_i, \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa\left(\tilde{t}_i\right)[\theta, t_{-i}]\, u_i\left(f\left(\tilde{t}_i, t_{-i}\right), \theta\right) \geq \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa\left(\tilde{t}_i\right)[\theta, t_{-i}]\, u_i\left(y\left(t_{-i}\right), \theta\right). \end{array} \right\} \quad (2)$$

Note that Y_i^f is a metrizable separable space.[14] We write Y^f for $\Pi_{i\in\mathcal{I}}Y_i^f$. For all $i \in \mathcal{I}$, let $Y_{i,s}^f$ be the set of all mappings in Y_i^f satisfying the inequality in (40) strictly for all $\tilde{t}_i \in T_i$.[15] Similarly, we write Y_s^f for $\Pi_{i\in\mathcal{I}}Y_{i,s}^f$.

Assumption 1 used by Oury and Tercieux (2012) to characterize a class of implementable SCFs can be stated as follows.

Definition 6 (Assumption 1 of Oury and Tercieux (2012))**.** Let $\mathcal{T}$ be any model and let $f : T \to \Delta(A)$ be any SCF. For all $i \in \mathcal{I}$, there exists $\bar{y}_i : T_{-i} \to \Delta(A)$ such that

[14]To see it, observe that $\Delta(A)$ is a separable metric space under the Prohokorov metric given that A is a separable metric space Aliprantis and Border (2006); Theorem 14.15). Moreover, a countable product of the space $\Delta(A)$ is separable metric space under the standard metric (see, e.g., Ok (2011), p. 196). Thus, since Y_i^f is a subset of a separable metric space, it follows that it is a separable metric space.

[15]Formally, for all $i \in \mathcal{I}$,

$$Y_{i,s}^f = \left\{ y : T_{-i} \to \Delta(A) \;\middle|\; \begin{array}{l} \text{For all } \tilde{t}_i \in T_i, \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa\left(\tilde{t}_i\right)[\theta, t_{-i}]\, u_i\left(f\left(\tilde{t}_i, t_{-i}\right), \theta\right) > \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa\left(\tilde{t}_i\right)[\theta, t_{-i}]\, u_i\left(y\left(t_{-i}\right), \theta\right). \end{array} \right\}$$

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for all $\psi_i \in \Delta(\Theta \times T_{-i})$, there exists $y_i \in Y_i^f$ satisfying

$$\begin{aligned} \textstyle\sum_{(\theta,t'_{-i})\in\Theta\times T_{-i}} \psi_i\left[\theta, t'_{-i}\right] u_i\left(y_i\left(t'_{-i}\right), \theta\right) \quad > \\ \textstyle\sum_{(\theta,t'_{-i})\in\Theta\times T_{-i}} \psi_i\left[\theta, t'_{-i}\right] u_i\left(\bar{y}_i\left(t'_{-i}\right), \theta\right). \end{aligned}$$

The assumption requires that player i's preferences over the mappings from T_{-i} to $\Delta(A)$ are such that there exists a mapping $\bar{y}_i : T_{-i} \to \Delta(A)$ such that, whatever his beliefs over $\Theta \times T_{-i}$ are, the mapping $\bar{y}_i$ is never his top-ranked mapping.

For the sake of clarity, in what follows, for every $i \in \mathcal{I}$, we write $T_{-i} \times \hat{T}_{-i}$ for $T_{-i} \times T_{-i}$. In the context of a mechanism, our interpretation of $\left(t_{-i}, \hat{t}_{-i}\right) \in T_{-i} \times \hat{T}_{-i}$ is that player i's opponents are of types t_{-i} but they are playing as if they were of types $\hat{t}_{-i}$.

A deception profile $\beta \in \mathcal{B}$ is *acceptable* on $\mathcal{T}$ for f if for all $t, t' \in T$, $t' \in \beta(t) \Longrightarrow f(t) = f(t')$. The following condition is due to Oury and Tercieux (2012). Before stating it, we need additional notation. For all $i \in \mathcal{I}$ and all $t_i \in T_i$, let $\Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ be defined by

$$\Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) = \left\{\nu_i \in \Delta\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) | \text{marg}_{\Theta \times T_{-i}} \nu_i = \kappa(t_i)\right\},$$

and, moreover, for all $\beta \in \mathcal{B}$, let $\Delta^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ be defined by

$$\Delta^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) = \left\{ \nu_i \;\middle|\; \begin{array}{l} \nu_i \in \Delta\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \text{ and} \\ \nu_i\left[\theta, t_{-i}, \hat{t}_{-i}\right] > 0 \Longrightarrow \hat{t}_{-i} \in \beta_{-i}(t_{-i}) \end{array} \right\}.$$

For the sake of brevity, we write $\Delta_{t_i}^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ for $\Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$.

Definition 7. $f : T \to \Delta(A)$ satisfies *interim (correlated) rationalizable monotonicity* (IRM, henceforth) on $\mathcal{T}$ if for every unacceptable deception profile $\beta \in \mathcal{B}$ on $\mathcal{T}$ for f, there exists $(i, t_i, t'_i) \in \mathcal{I} \times T_i \times \beta_i(t_i)$ such that for all $\psi_i(t_i) \in$

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$\Delta_{t_i}^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$, there exists $y_i^* \in Y_i^f$ such that

$$\begin{array}{ll}\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}}\left(\text{marg}_{\Theta\times\hat{T}_{-i}}\psi_i\left(t_i\right)\left[\theta,\hat{t}_{-i}\right]\right)u_i\left(y_i^*\left(\hat{t}_{-i}\right),\theta\right) & > \\ \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}}\left(\text{marg}_{\Theta\times\hat{T}_{-i}}\psi_i\left(t_i\right)\left[\theta,\hat{t}_{-i}\right]\right)u_i\left(f\left(t_i',\hat{t}_{-i}\right),\theta\right). & \end{array} \tag{3}$$

Remark 2. Bergemann and Morris (2008b) and Oury and Tercieux (2012) introduce a strict variant of IRM. f satisfies strict IRM on $\mathcal{T}$ if $y^* \in Y_i^f$ satisfying (3) is such that the inequality (40) holds strictly if $t_i' = \tilde{t}_i$. However, IRM is equivalent to strict IRM. We need only discuss the implication that IRM implies strict IRM. To this end, observe that Proposition 2 of Bergemann and Morris (2008b) shows that any function that is ICR-implementable by a finite mechanism satisfies strict IRM. The arguments of the proof of Proposition 2 can be adapted to show that only strict interim rationalizable monotonic functions are ICR-implementable by a mechanism satisfying the BRP of Bergemann and Morris (2008b). Since Theorem 1 below shows that IRM is sufficient for interim rationalizable (and Bayes-Nash) implementation, it follows that IRM implies strict IRM.[16]

A condition, which is equivalent to IRM, can be expressed in terms of the limit point of an iterative net of deception profiles. The iterative net, denoted by $(\beta^\alpha)_{\alpha\in\Omega}$, is defined as follows. For all $i \in \mathcal{I}$ and all $t_i \in T_i$, let

$$\beta_i^0\left(t_i\right) = \bar{\beta}_i\left(t_i\right) = T_i,$$

and, for all ordinal numbers $\alpha \in \Omega$, define $\beta_i^\alpha\left(t_i\right)$ as follows:

[16]This equivalence is reminiscent of the equivalence between strict Maskin monotonicity and Maskin monotonicity under the no worst alternative property (see Bergemann et al. (2010), footnote 5, p. 1261). Using this insight, Jain et al. (2021)) show that Strict Event Monotonicity**, due to Xiong (2021)), is equivalent to Event Monotonicity**. Strict Event Monotonicity** fully characterizes the class of SCFs that are rationalizably implementable under compete information when there are three or more players (see p. xx of Jain et al. (2021) for further details.).

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- If α is a successor ordinal, then

$$\beta_i^{\alpha}(t_i) = \left\{ \hat{t}_i \;\middle|\; \begin{array}{l} \hat{t}_i \in \beta_i^{\alpha-1}(t_i) \text{ and there exists} \\ \nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^{\alpha-1}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \text{ such that} \\ \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \left(\text{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) \geq \\ \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \left(\text{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(y_i\left(\hat{t}_{-i}\right),\theta\right), \\ \text{for all } y_i \in Y_i^f. \end{array} \right\} \tag{4}$$

- If α is a limit ordinal, then

$$\beta_i^{\alpha}(t_i) = \bigcap_{\gamma<\alpha} \beta_i^{\gamma}(t_i). \tag{5}$$

Observe that $t_i \in \beta_i^{\alpha}(t_i)$ for all $i \in \mathcal{I}$, all $t_i \in T_i$ and all $\alpha \in \Omega$. A net $(\beta^{\alpha})_{\alpha\in\Omega}$ is monotonic decreasing if $\beta^{\alpha+1} \subseteq \beta^{\alpha}$ for all $\alpha \in \Omega$. If the limit of $(\beta^{\alpha})_{\alpha\in\Omega}$ exists, we denote it by β^*; that is, $\lim_{\alpha\in\Omega}\beta^{\alpha} \to \beta^*$.

Lemma 3. Let $\mathcal{T}$ be any model. $(\beta^{\alpha})_{\alpha\in\Omega}$ is a monotonic decreasing net such that $\lim_{\alpha\in\Omega}\beta^{\alpha} \to \beta^* \in \mathcal{B}$. Moreover, there exists an ordinal $\alpha \in \Omega$ such that $\beta^{\alpha} = \beta^{\alpha+1} = \beta^*$.

Proof. Let $\mathcal{T}$ be any model. Let $(\beta^{\alpha})_{\alpha\in\Omega}$ be given. By definition $(\beta^{\alpha})_{\alpha\in\Omega}$, it holds that $\beta^t \subseteq \beta^{\alpha}$ for all $\alpha \in\Omega$. Thus, $\beta^{\alpha} \in \mathcal{B}$ for all $\alpha \in\Omega$ and it is bounded below. Moreover, since $(\beta^{\alpha})_{\alpha\in\Omega}$ is bounded below, it holds that $\lim_{\alpha\in\Omega}\beta^{\alpha} \to \beta^* \in \mathcal{B}$ if it is a monotonic decreasing net. Thus, we show that $(\beta^{\alpha})_{\alpha\in\Omega}$ is a monotonic decreasing net. Fix any ordinal $\alpha \in\Omega$. Fix any $i \in \mathcal{I}$ and any $t_i \in T_i$. We show that $\beta_i^{\alpha+1}(t_i) \subseteq \beta_i^{\alpha}(t_i)$. Let us proceed according to whether α is a successor ordinal or not.

- Suppose that α is a successor ordinal. Fix any $\hat{t}_i \in \beta_i^{\alpha+1}(t_i)$. We show that $\hat{t}_i \in \beta_i^{\alpha}(t_i)$. (4) implies that $\hat{t}_i \in \beta_i^{\alpha}(t_i)$, as we sought.

- Suppose that α is a limit ordinal. Since α is a limit ordinal, it follows that $\alpha+1$ is a successor ordinal. Suppose that $\hat{t}_i \in \beta_i^{\alpha+1}(t_i)$. We show that $\hat{t}_i \in \beta_i^{\alpha}(t_i)$. Again, (4) implies that $\hat{t}_i \in \beta_i^{\alpha}(t_i)$, as we sought.

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Since the choice of α was arbitrary, it follows that $\lim_{\alpha\in\Omega}\beta^{\alpha}\rightarrow\beta^{*}\in\mathcal{B}$. Finally, the fact that there exists an ordinal $\alpha\in\Omega$ such that $\beta^{\alpha}=\beta^{\alpha+1}=\beta^{*}$ follows from the assumption that T_i is a countable set for each player i and the fact that Ω is an uncountable set. To see this, fix any $i\in\mathcal{I}$ and any $t_i\in T_i$. Assume that, for all $\alpha\in\Omega$, it holds that $\beta_i^{\alpha+1}(t_i)\subset\beta_i^{\alpha}(t_i)$.[17] Define the mapping $f:\Omega\rightarrow T_i$ by $f(\alpha)\in\beta_i^{\alpha}(t_i)\setminus\beta_i^{\alpha+1}(t_i)$, for all $\alpha\in\Omega$. Let us show that f is an injective mapping. Fix any $\alpha,\alpha'\in\Omega$ such that $\alpha\neq\alpha'$. Let us show that $f(\alpha)\neq f(\alpha')$. Since Ω is a well ordered set, it is Without loss of generality, let $\alpha'>\alpha$. Since $\beta_i^{\alpha'}(t_i)\subseteq\beta_i^{\alpha+1}(t_i)\subset\beta_i^{\alpha}(t_i)$, it follows from definition of f that $f(\alpha)\neq f(\alpha')$. Since f is an injective mapping from Ω to T_i, it follows that Ω is a countable set, which is a contradiction. Thus, for all $i\in\mathcal{I}$, all $t_i\in T_i$, there exists $\alpha\in\Omega$ such that $\beta_i^{\alpha+1}(t_i)=\beta_i^{\alpha}(t_i)$. Since Ω is an uncountable set whose elements are ordered by $\geq$, it follows that there exists $\alpha\in\Omega$ such that $\beta^{\alpha+1}=\beta^{\alpha}$. □

Our condition can be stated as follows.

Definition 8. $f:T\rightarrow\Delta(A)$ satisfies *Interim Iterative Monotonicity* (IIM, henceforth) on $\mathcal{T}$ if β^{*} is an acceptable deception on $\mathcal{T}$ for f.

The following result shows that IIM is equivalent to IRM.

Lemma 4. $f:T\rightarrow\Delta(A)$ satisfies IIM on $\mathcal{T}$ if and only if f satisfies IRM on $\mathcal{T}$.

Proof. Assume that $f:T\rightarrow\Delta(A)$ satisfies IIM on $\mathcal{T}$. Take any unacceptable deception profile $\beta\in\mathcal{B}$ on $\mathcal{T}$ for f. Assume, to the contrary, that for all $(i,t_i,t_i')\in\mathcal{I}\times T_i\times\beta_i(t_i)$, there exists $\psi_i(t_i)\in\Delta_{t_i}^{\beta_{-i}}\left(\Theta\times T_{-i}\times\hat{T}_{-i}\right)$ such that for all $y_i^{*}\in Y_i^{f}$, (3) is violated.[18] To derive a contradiction, let us first show that $\beta\subseteq\beta^{\alpha}$ for all $\alpha\in\Omega$. Let us proceed by transfinite induction.

By definition, $\beta\subseteq\bar{\beta}=\beta^{0}$. Fix an arbitrary $\alpha\in\Omega$ and suppose that for all $\gamma<\alpha$, it holds that $\beta\subseteq\beta^{\gamma}$. To complete the proof we need to show that $\beta\subseteq\beta^{\alpha}$. We proceed according to whether α is a limit ordinal or a successor ordinal. When

[17]The symbol $\subset$ denotes strict set inclusion.

[18]Recall that Y^{f} is a nonempty metrizable subspace.

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α is a limit ordinal, the induction hypothesis and the definition of β^α implies that $\beta \subseteq \bigcap_{\gamma<\alpha} \beta^\gamma = \beta^\alpha$.

Suppose that α is a successor ordinal. Let us show that $\beta \subseteq \beta^\alpha$. By the inductive hypothesis, it holds that $\psi_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^{\alpha-1}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ for all $i \in \mathcal{I}$ and all $t_i \in T_i$. Fix any $i \in \mathcal{I}$ and any $t_i \in T_i$. Take any $\hat{t}_i \in \beta_i(t_i)$. It follows from the inductive hypothesis that $\hat{t}_i \in \beta_i^{\alpha-1}(t_i)$. Since (3) is violated for $y_i^* \in Y_i^f$, (4) implies that $\hat{t}_i \in \beta_i^\alpha(t_i)$. Since the triplet $\left(i, t_i, \hat{t}_i\right) \in \mathcal{I} \times T_i \times \beta_i(t_i)$ was chosen arbitrarily, we conclude that $\beta \subseteq \beta^\alpha$. By the principle of transfinite induction, it holds that $\beta \subseteq \beta^\alpha$ for all $\alpha \in \Omega$. Since Lemma 3 implies that the $(\beta^\alpha)_{\alpha\in\Omega}$ is a monotonically decreasing net which converges to $\beta^* \in \mathcal{B}$, we have that $\beta \subseteq \beta^*$. Since f satisfies IIM on $\mathcal{T}$, it follows that β^* is an acceptable deception profile on $\mathcal{T}$ for f, and so β is also an acceptable deception profile on T for f, which is a contradiction.

Assume f satisfies IRM on $\mathcal{T}$. Assume, to the contrary, that $\beta^* \in \mathcal{B}$ is not acceptable on $\mathcal{T}$ for f. Since f satisfies IRM, it follows that there exists $(i, t_i, t_i') \in \mathcal{I} \times T_i \times \beta_i^*(t_i)$ such that for all $\psi_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^*}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$, there exists $y_i^* \in Y_i^f$ such that (3) is satisfied. Lemma 3 implies that there exists an $\alpha \in \Omega$ such that $\beta^\alpha = \beta^{\alpha+1} = \beta^*$. Since $t_i' \in \beta_i^*(t_i)$, (4) implies that there exists $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^*}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ such that

$$\begin{aligned}\textstyle\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}}\left(\operatorname{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right)u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) &\geq \\ \textstyle\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}}\left(\operatorname{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right)u_i\left(y_i^*\left(\hat{t}_{-i}\right),\theta\right)\end{aligned}$$

for all $y_i^* \in Y_i^f$, yielding a contradiction. □

Any SCF satisfying our condition on $\mathcal{T}$ is *incentive compatible* on $\mathcal{T}$. The condition can be stated as follows.

Definition 9. $f : T \to \Delta(A)$ *incentive compatible* on $\mathcal{T}$ if for all $i \in \mathcal{I}$ and all $t_i \in T_i$,

$$\sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(t_i)[\theta, t_{-i}]\, u_i(f(t_i, t_{-i}), \theta) \geq \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(t_i)[\theta, t_{-i}]\, u_i(f(t_i', t_{-i}), \theta)$$

for all $t_i \in T_i$.

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Lemma 5. $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$ implies that f is incentive compatible on $\mathcal{T}$.

Proof. It follows from Lemma 4 above and Lemma 3 of Oury and Tercieux (2012). $\square$

V. A full characterization

Our main result can be stated as follows.

Theorem 1. The following statements are equivalent.

(i) $f : T \to \Delta(A)$ is ICR-implementable on $\mathcal{T}$ by a mechanism satisfying the EBRP.

(ii) $f : T \to \Delta(A)$ satisfies IRM on $\mathcal{T}$.

(iii) $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$.

(iv) $f : T \to \Delta(A)$ is both ICR-implementable and Bayes–Nash implementable on $\mathcal{T}$.

Proof of Theorem 1

The proof that part (i) implies part (ii) can be found in Appendix A. Lemma 4 implies that part (ii) implies (iii). Lemma 2 implies that part (iv) implies (i). Finally, we show that part (iii) implies part (iv). Thus, assume that $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$. We show that $f : T \to \Delta(A)$ is ICR-implementable on $\mathcal{T}$ by a mechanism satisfying the EBRP. Before proving this result, we need additional notation. Fix any $\beta \in \mathcal{B}$, any $i \in \mathcal{I}$, and any $t_i \in T_i$. Let $\Delta_{t_i}^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$ be defined by

$$\Delta_{t_i}^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right) = \left\{ \psi_i \;\middle|\; \begin{array}{l} \text{There exists } \nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \\ \text{such that } \operatorname{marg}_{\Theta \times \hat{T}_{-i}} \nu_i(t_i) = \psi_i. \end{array} \right\} \tag{6}$$

The following definition is critical in the construction of our implementing mechanism.

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Definition 10. Let $\mathcal{T}$ be any model. For all $\beta \in \mathcal{B}$ and all $i \in \mathcal{I}$, $t_i \in T_i(\beta)$ if and only if for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, there exist $y_i, \bar{y}_i \in Y_i^f$ such that

$$\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(y_i\left(\hat{t}_{-i}\right),\theta\right) > \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right). \quad (7)$$

The above definition says that type $t_i \in T_i(\beta)$ provided that for each belief ψ_i of type t_i over $\Theta \times \hat{T}_{-i}$, there are mappings $y_i, \bar{y}_i \in Y_i^f$, which may depend on his belief ψ_i, such that y_i is strictly preferred to $\bar{y}_i$, given his belief ψ_i. A stronger, though more desirable, definition would be to require that the mapping $\bar{y}_i$ does not depend on player i's belief. The definition can be stated as follows.

Definition 11. Let $\mathcal{T}$ be any model. For all $\beta \in \mathcal{B}$ and all $i \in \mathcal{I}$, $t_i \in T_i^*(\beta)$ if and only if there exist $\bar{y}_i \in Y_i^f$ such that for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, there exists $y_i \in Y_i^f$ such that

$$\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(y_i\left(\hat{t}_{-i}\right),\theta\right) > \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right). \quad (8)$$

Observe that Definition 11 is implied by Assumption 1 of Oury and Tercieux (2012) when $\beta = \bar{\beta}$. Moreover, Definition 10 is implied by the no-worst-rule condition of Kunimoto (2019) when $\beta = \bar{\beta}$.

By adapting the arguments of Lemma 6 of Jain et al. (2021) to interim set up, we show below that Definition 11 and Definition 10 are equivalent.

Lemma 6. Let $\mathcal{T}$ be any model. For all $\beta \in \mathcal{B}$, $T^*(\beta) = T(\beta)$.

Proof. Let $\mathcal{T}$ be any model. Fix any $\beta \in \mathcal{B}$ and $i \in \mathcal{I}$. Since it is clear that $T_i^*(\beta) \subseteq T_i(\beta)$, let us show that $T_i(\beta) \subseteq T_i^*(\beta)$. Assume that $t_i \in T_i(\beta)$. Definition 10 implies that for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, there exist $y_i^{\psi_i}, \bar{y}_i^{\psi_i} \in Y_i^f$ such that (7) is satisfied. Since $\Delta_{t_i}^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$ is a separable metric space, let $\hat{\Delta}\left(\Theta \times \hat{T}_{-i}\right) = \cup_{k\in\mathbb{N}}\left\{\psi_{i,k}\right\}$ be

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a countable, dense subset of $\Delta_{t_i}^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$. Let $\tilde{y}_i \in Y_i^f$ be a mapping defined by

$$\tilde{y}_i = \sum_{k=1}^{\infty} \frac{1}{2^k} \bar{y}_i^{\psi_{i,k}}.$$

For all $\bar{k} \in \mathbb{N}$, let $y_i^{\psi_{i,\bar{k}}} \in Y_i^f$ be a mapping defined by

$$y_i^{\bar{k}} = \sum_{k \neq \bar{k}} \frac{1}{2^k} \bar{y}_i^{\psi_{i,k}} + \frac{1}{2^{\bar{k}}} y_i^{\psi_{i,\bar{k}}}.$$

Thus, for all $k \in \mathbb{N}$, we have that

$$\sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \psi_{i,k}\left[\theta, \hat{t}_{-i}\right] u_i\left(y_i^k\left(\hat{t}_{-i}\right), \theta\right) > \sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \psi_i\left[\theta, \hat{t}_{-i}\right] u_i\left(\tilde{y}_i\left(\hat{t}_{-i}\right), \theta\right),$$

where the strict inequality is guaranteed by (7). Since player i's preference over lotteries are continuous and since, moreover, $\hat{\Delta}\left(\Theta \times \hat{T}_{-i}\right)$ is a countable, dense subset of $\bigcup_{t_i \in T_i} \Delta_{t_i}^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, it follows that $i \in \mathcal{I}^*(\beta)$. Since the choice of $i \in \mathcal{I}(\beta)$ was arbitrary, it follows that $T_i(\beta) \subseteq T_i^*(\beta)$. □

Lemma 7. Assume that $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$. For all $\alpha \in \Omega$, all $i \in \mathcal{I}$ and all $t_i \in T_i$, $t_i \in T_i^c(\beta^\alpha) \implies \beta_i^\alpha(t_i) = \beta_i^{\alpha+1}(t_i) = \bar{\beta}_i(t_i)$.[19]

Proof. Assume that $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$. Fix any $\alpha \in \Omega$. Assume that $t_i \in T_i^c(\beta^\alpha)$. Assume, to the contrary, that $\beta_i^{\alpha+1}(t_i) \neq \beta_i^\alpha(t_i)$. Since Lemma 3 implies that $(\beta^\alpha)_{\alpha \in \Omega}$ is a monotonic decreasing net, it follows that there exists $\left(t_i, \hat{t}_i\right)$ such that $\hat{t}_i \in \beta_i^\alpha(t_i)$ and $\hat{t}_i \notin \beta_i^{\alpha+1}(t_i)$. It follows from (4) that for all $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^\alpha}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$,

$$\begin{aligned}&\textstyle\sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \left(\operatorname{marg}_{\Theta \times \hat{T}_{-i}} \nu_i(t_i)\left[\theta, \hat{t}_{-i}\right]\right) u_i\left(f\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right) \quad < \\ &\textstyle\sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \left(\operatorname{marg}_{\Theta \times \hat{T}_{-i}} \nu_i(t_i)\left[\theta, \hat{t}_{-i}\right]\right) u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right), \theta\right)\end{aligned}$$

[19] $T_i^c(\beta^\alpha) = \{t_i \in T_i | t_i \notin T_i(\beta^\alpha)\}$.

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for some $\bar{y}_i \in Y_i^f$. Therefore, for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}^{\alpha}}\left(\Theta \times \hat{T}_{-i}\right)$,

$$\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) < \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right) \quad (9)$$

for some $\bar{y}_i \in Y_i^f$. Let $y_i(\cdot) = f\left(\hat{t}_i,\cdot\right)$. Since f satisfies IIM on $\mathcal{T}$, Lemma 4 and Lemma 18 imply that f is incentive compatible on $\mathcal{T}$. This implies that $y_i(\cdot) \in Y_i^f$. We have that the inequality in (7) holds for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}^{\alpha}}\left(\Theta \times \hat{T}_{-i}\right)$. Definition 10 implies that $t_i \in T_i(\beta^\alpha)$, which is a contradiction.

Finally, let us show that $\beta_i^{\alpha+1}(t_i) = \beta_i^{\alpha}(t_i) = \bar{\beta}_i(t_i)$. Assume, to the contrary, that $\beta_i^{\alpha}(t_i) \neq \bar{\beta}_i(t_i)$. Since Lemma 3 implies that $\left(\beta_i^{\alpha}\right)_{\alpha\in\Omega}$ is a decreasing monotonic net, it follows that there exists a successor ordinal $\hat{\alpha}$ such that $0 < \hat{\alpha} \leq \alpha$ and that $\beta_i^{\hat{\alpha}}(t_i) \subseteq \beta_i^{\hat{\alpha}-1}(t_i)$ and $\beta_i^{\hat{\alpha}}(t_i) \neq \beta_i^{\hat{\alpha}-1}(t_i)$.[20] Thus, $\hat{t}_i \in \beta_i^{\hat{\alpha}-1}(t_i)$ and $\hat{t}_i \notin \beta_i^{\hat{\alpha}}(t_i)$ for some $\hat{t}_i \in T_i$. (4) implies that there exists $\bar{y}_i \in Y_i^f$ such that

$$\begin{array}{l}\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}}\left(\operatorname{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) \quad < \\ \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}}\left(\operatorname{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)\end{array}$$

for all $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^{\hat{\alpha}-1}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$. By definition of $\Delta^{\beta_{-i}^{\hat{\alpha}-1}} t_i\left(\Theta \times \hat{T}_{-i}\right)$ in (6), it follows that there exists $\bar{y}_i \in Y_i^f$ such that

$$\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) < \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)$$

for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}^{\hat{\alpha}-1}}\left(\Theta \times \hat{T}_{-i}\right)$. Let $y_i(\cdot) = f\left(\hat{t}_i,\cdot\right)$. Since f satisfies IIM on $\mathcal{T}$,

[20] Suppose not. Then, for all successor ordinals $\hat{\alpha}$ such that $\hat{\alpha} \leq \alpha$, it holds that $\beta_i^{\hat{\alpha}}(t_i) = \beta_i^{\hat{\alpha}-1}(t_i)$. Suppose that $\beta_i^{\hat{\alpha}}(t_i) = \bar{\beta}_i(t_i)$ for all successor ordinals $\hat{\alpha}$ such that $\hat{\alpha} \leq \alpha$. It follows that for every limit ordinal $\delta \leq \alpha$, it holds that $\beta_i^{\delta}(t_i) = \bigcap_{\gamma<\delta}\beta_i^{\gamma}(t_i) = \bar{\beta}_i(t_i)$. An immediate contradiction is obtain if α is a limit ordinal. Thus, let α be a successor ordinal, and so $\beta_i^{\alpha}(t_i) = \bar{\beta}_i(t_i)$, which is a contradiction. Thus, there exists a successor ordinal α', with $\alpha' \leq \alpha$, such that $\beta_i^{\alpha'}(t_i) \neq \bar{\beta}_i(t_i)$. Since for all successor ordinals $\hat{\alpha}$ such that $\hat{\alpha} \leq \alpha$, it holds that $\beta_i^{\hat{\alpha}}(t_i) = \beta_i^{\hat{\alpha}-1}(t_i)$, it follows that for all successor ordinals $\hat{\alpha}$ such that $\hat{\alpha} \leq \alpha$, it holds that $\beta_i^{\hat{\alpha}}(t_i) \neq \bar{\beta}_i(t_i)$. Since $1 \in \Omega$ is a successor ordinal, it follows that there exists a successor ordinal such that $\beta_i^{1}(t_i) \subseteq \beta_i^{0}(t_i) = \bar{\beta}_i(t_i)$, yielding a contradiction.

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Lemma 4 and Lemma 18 imply that f is incentive compatible on $\mathcal{T}$. This implies that $y_i(\cdot) \in Y_i^f$. Thus, Definition 10 implies that $t_i \in T_i\left(\beta^{\hat{\alpha}-1}\right)$. Since Lemma 3 implies that $(\beta_i^\alpha)_{\alpha\in\Omega}$ is a decreasing monotonic sequence and since, moreover, $\hat{\alpha}$ is such that $0 \neq \hat{\alpha} \leq \alpha$, it follows that

$$\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) < \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)$$

for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}^{\alpha}}\left(\Theta\times\hat{T}_{-i}\right) \subseteq \Delta_{t_i}^{\beta_{-i}^{\hat{\alpha}-1}}\left(\Theta\times\hat{T}_{-i}\right)$. Definition 10 implies that $t_i \in T_i(\beta^\alpha)$, which is a contradiction. Thus, $\beta_i^{\alpha+1}(t_i) = \beta_i^\alpha(t_i) = \bar{\beta}_i(t_i)$. □

Lemma 8. Assume that $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$.

(i) If $T_i\left(\bar{\beta}\right) = \emptyset$ for all $i \in \mathcal{I}$, then f is constant.[21]

(ii) For all $i \in \mathcal{I}$, If $T_i(\beta^*) \neq T_i$, then for all $t_{-i} \in T_{-i}$ and all $t_i, t_i' \in T_i$, $f(t_i,t_{-i}) = f(t_i',t_{-i})$.

Proof. Assume that $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$. To show part (i), assume that $T_i\left(\bar{\beta}\right) = \emptyset$ for all $i \in \mathcal{I}$. Let us proceed by transfinite induction. We show that for all $i \in \mathcal{I}$, $\beta_i^\alpha(t_i) = \bar{\beta}_i(t_i)$ for all $t_i \in T_i$. The statement is clearly true for all $i \in \mathcal{I}$ if $\alpha = 0$. Thus, let $\alpha \neq 0$.

Suppose that α is a successor ordinal. Suppose that the statement is true for $\alpha - 1$ for all $i \in \mathcal{I}$ and all $t_i \in T_i$. Fix any $i \in \mathcal{I}$ and any $t_i \in T_i$. Assume, to the contrary, that $\beta_i^\alpha(t_i) \neq \bar{\beta}_i(t_i) = T_i$. Then, there exists $\hat{t}_i \in T_i$ such that $\hat{t}_i \notin \beta_i^\alpha(t_i)$ and $\hat{t}_i \in \beta_i^{\alpha-1}(t_i) = \bar{\beta}_i(t_i)$. It follows from (4) that for all $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^{\alpha-1}}\left(\Theta\times T_{-i}\times\hat{T}_{-i}\right)$,

$$\begin{aligned}&\textstyle\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}}\left(\operatorname{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right)u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) \quad < \\ &\textstyle\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}}\left(\operatorname{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right)u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)\end{aligned}$$

for some $\bar{y}_i \in Y_i^f$. By definition of $\Delta_{t_i}^{\beta_{-i}^{\alpha-1}}\left(\Theta\times\hat{T}_{-i}\right)$ in (6), it follows that there exists

[21] f is constant if for all $t, t' \in T$, $f(t) = f(t')$.

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$\bar{y}_i \in Y_i^f$ such that

$$\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) < \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)$$

for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}^{\alpha-1}}\left(\Theta\times\hat{T}_{-i}\right)$. Let $y_i(\cdot) = f\left(\hat{t}_i,\cdot\right)$. Since f satisfies IIM on $\mathcal{T}$, Lemma 4 and Lemma 18 imply that f is incentive compatible on $\mathcal{T}$. This implies that $y_i(\cdot) \in Y_i^f$. Since the choice of $t_i \in T_i$ was arbitrary and since $\beta_{-i}^{\alpha-1}(t_{-i}) = \bar{\beta}_{-i}(t_{-i})$ for all $t_{-i} \in T_{-i}$, we have that $t_i \in T_i\left(\bar{\beta}\right)$, which is a contradiction. Thus, we conclude that $\beta_i^\alpha(t_i) = \bar{\beta}_i(t_i)$ for all $i \in \mathcal{I}$ and all $t_i \in T_i$.

Suppose that $\alpha \neq 0$ is a limit ordinal. Suppose that for all $i \in \mathcal{I}$ and all $t_i \in T_i$, it holds that $\beta_i^\gamma(t_i) = \bar{\beta}_i(t_i)$. Since, by definition, $\beta_i^\alpha(t_i) = \bigcap_{\gamma<\alpha} \beta_i^\gamma(t_i)$ for all $i \in \mathcal{I}$ and all $t_i \in T_i$, it follows that $\beta_i^\alpha(t_i) = \bar{\beta}_i(t_i)$ for all $i \in \mathcal{I}$ and all $t_i \in T_i$.

Since β^* is the limit point of $(\beta^\alpha)_{\alpha\in\Omega}$, it follows that $\beta_i^*(t_i) = \bar{\beta}_i(t_i)$ for all $i \in \mathcal{I}$ and all $t_i \in T_i$. Fix any $t^* \in T$. Since f satisfies IIM on $\mathcal{T}$, it follows that for all $t \in \beta^*(t^*) = T$, it holds that $f(t) = f(t^*)$. Thus, f is constant. This completes the proof of part (i).

Let us show part (*ii*). Fix any $i \in \mathcal{I}$ such that $T_i(\beta^*) \neq T_i$. Let us show that $f(t_i,t_{-i}) = f(t_i',t_{-i})$ for all $t_i,t_i' \in T_i$ and $t_{-i} \in T_{-i}$. Since f satisfies IIM on $\mathcal{T}$, it is enough to show that $\beta_i^*(t_i) = T_i$ for some $t_i \in T_i$.[22] Assume, to the contrary, that $\beta_i^*(t_i) \neq T_i$ for all $t_i \in T_i$. Fix any successor ordinal α such that $\beta_i^*(t_i) = \beta_i^\alpha(t_i) = \beta_i^{\alpha-1}(t_i)$ for all $t_i \in T_i$. It follows from (4) that for all $t_i \in T_i$ and all $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^{\alpha-1}}\left(\Theta\times T_{-i}\times\hat{T}_{-i}\right)$,

$$\begin{array}{l}\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}}\left(\mathrm{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right)u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) \quad < \\ \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}}\left(\mathrm{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right)u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)\end{array}$$

for some $\bar{y}_i \in Y_i^f$. By definition of $\Delta_{t_i}^{\beta_{-i}^{\alpha-1}}\left(\Theta\times\hat{T}_{-i}\right)$ in (6), it follows that there exists

[22]To see it, suppose that $\beta^*(t_i) = T_i$ for some $t_i \in T_i$. Fix any $t_{-i} \in T_i$. Since β^* is an acceptable deception, it follows that $f(t_i',t_{-i}) = f(t_i'',t_{-i})$ for all $t_i',t_i'' \in \beta_i^*(t_i)$.

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$\bar{y}_i \in Y_i^f$ such that

$$\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) < \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)$$

for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}^{\alpha-1}}\left(\Theta\times\hat{T}_{-i}\right)$ and all $t_i \in T_i$. Let $y_i(\cdot) = f\left(\hat{t}_i,\cdot\right)$. Since f satisfies IIM on $\mathcal{T}$, Lemma 4 and Lemma 18 imply that f is incentive compatible on $\mathcal{T}$. This implies that $y_i(\cdot) \in Y_i^f$. Definition 10 implies that $T_i(\beta^*) = T_i$, which is a contradiction. This completes the proof of part (ii). □

In what follows, to avoid trivialities, we assume that $T_i\left(\bar{\beta}\right) \neq \emptyset$ for some $i \in \mathcal{I}$. Moreover, we assume that $T_i(\beta^*) = T_i$ for all $i \in \mathcal{I}$.

Lemma 9. For all $(\alpha, i, t_i) \in \Omega\times\mathcal{I}\times T_i$, if $t_i \in T_i(\beta^\alpha)\backslash T_i(\beta^0)$, then there exists $\hat{\alpha}(t_i) \leq \alpha$ such that $t_i \in T_i\left(\beta^{\hat{\alpha}(t_i)}\right)$ and $t_i \in T_i^c(\beta^\gamma)$ for all $\gamma < \hat{\alpha}(t_i)$.

Proof. Fix any $(\alpha, i, t_i) \in \Omega\times\mathcal{I}\times T_i$ such that $t_i \in T_i(\beta^\alpha)\backslash T_i(\beta^0)$. Assume, to the contrary, that there does not exist any $\hat{\alpha} \in \Omega$, with $\hat{\alpha} \leq \alpha$, such that $t_i \in T_i\left(\beta^{\hat{\alpha}}\right)$ and $t_i \in T_i^c(\beta^\gamma)$ for all $\gamma < \hat{\alpha}$. Thus, for all $\hat{\alpha} \in \Omega$, with $\hat{\alpha} \leq \alpha$, it holds that $t_i \in T_i^c\left(\beta^{\hat{\alpha}}\right)$ or $t_i \in T_i(\beta^\gamma)$ for some $\gamma < \hat{\alpha}$.

Suppose that there exists $\hat{\alpha} \in \Omega$, with $\hat{\alpha} \leq \alpha$, such that $t_i \in T_i(\beta^\gamma)$ for some $\gamma < \hat{\alpha}$. Let us consider the set $\bar{\Omega} = \left\{\delta \in \Omega\backslash\{0\} \,|\, \delta \leq \gamma < \hat{\alpha} \text{ and } t_i \in T_i\left(\beta^\delta\right)\right\}$. Let $\gamma^* \in \bar{\Omega}$ be such that $\gamma^* \leq \delta$ for all $\delta \in \bar{\Omega}$. We have that $t_i \in T_i\left(\beta^{\gamma^*}\right)$ and $t_i \in T_i^c(\beta^\gamma)$ for all $\gamma < \gamma^*$, which is a contradiction. Otherwise, suppose that for all $\hat{\alpha} \in \Omega$, with $\hat{\alpha} \leq \alpha$, $t_i \in T_i^c(\beta^\gamma)$ for all $\gamma < \hat{\alpha}$. Since $t_i \in T_i(\beta^\alpha)$, it follows that $t_i \in T_i(\beta^\alpha)$ and $t_i \in T_i^c(\beta^\gamma)$ for all $\gamma < \alpha$, which is a contradiction. □

The following result is useful in defining *Rule 3* of the mechanism.

Lemma 10. Let $\mathcal{T}$ be any model. For all $i \in \mathcal{I}$ such that $T_i(\beta^*) \neq \emptyset$, there exists $\hat{y}_i \in \Delta(A)$ such that for every $\phi_i \in \Delta(\Theta)$, there exists $y_i \in \Delta(A)$ such that

$$\sum_{\theta\in\Theta} \phi_i(\theta)\, u_i(y_i,\theta) > \sum_{\theta\in\Theta} \phi_i(\theta)\, u_i(\hat{y}_i,\theta). \tag{10}$$

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Proof. Fix an $i \in \mathcal{I}$ and $t_i \in T_i(\beta^*)$. Lemma 6 implies that $t_i \in T_i^*(\beta^*)$. Definition 11 implies that there exists $\bar{y}_i \in Y_i^f$ such that for all $\psi_i \in \Delta_{t_i}^{\beta^*_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, there exists $y_i \in Y_i^f$ such that (8) holds. Since $\beta^t \subseteq \beta^*$, it follows that there exists $\bar{y}_i \in Y_i^f$ such that for all $\psi_i \in \Delta_{t_i}^{\beta^t_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, there exists $y_i \in Y_i^f$ such that (8) holds. Fix any $t_i \in T_i$. Observe that $\phi_i \circ \left(\text{marg}_{T_{-i}}\kappa(t_i)\right) \in \Delta_{t_i}^{\beta^t_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$ for all $\phi_i \in \Delta(\Theta)$. Therefore, it holds that

$$\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \left(\phi_i[\theta]\left(\text{marg}_{T_{-i}}\kappa(t_i)\left[\hat{t}_{-i}\right]\right)\right)\left[u_i\left(y_i\left(\hat{t}_{-i}\right),\theta\right) - u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)\right] > 0.$$

By setting

$$y_i = \sum_{\hat{t}_{-i}\in\hat{T}_{-i}} \left(\text{marg}_{T_{-i}}\kappa(t_i)\left[\hat{t}_{-i}\right]\right) y_i\left(\hat{t}_{-i}\right)$$

and

$$\hat{y}_i = \sum_{\hat{t}_{-i}\in\hat{T}_{-i}} \left(\text{marg}_{T_{-i}}\kappa(t_i)\left[\hat{t}_{-i}\right]\right) \bar{y}_i\left(\hat{t}_{-i}\right),$$

and by noting that $y_i, \hat{y}_i \in \Delta(A)$, the inequality in (10) follows for i. Since the choice of $i \in \mathcal{I}$ such that $T_i(\beta^*) \neq \emptyset$ was arbitrary, the statement follows. □

Let $\mathcal{T}$ be any model. Since $T_i(\beta^*) = T_i$ for all $i \in \mathcal{I}$ and since Lemma 10 guarantees the existence of the lottery $\hat{y}_i \in \Delta(A)$ for all $i \in \mathcal{I}$, let us define the lottery $\hat{y}$ by

$$\hat{y} = \frac{1}{I}\sum_{i\in\mathcal{I}} \hat{y}_i.$$

Given the net $(\beta^\alpha)_{\alpha\in\Omega}$ and our assumption that $T(\beta^*) = T$, Lemma 3 implies that for some $\alpha \in \Omega$, it holds that $T(\beta^\alpha) = T$. Fix any $i \in \mathcal{I}$ and any $t_i \in T_i$. Then, if $t_i \in T_i \setminus T_i(\beta^0)$, Lemma 9 implies that there exists a least ordinal $\alpha(t_i)$ such that $t_i \in T_i\left(\beta^{\alpha(t_i)}\right) \setminus T_i(\beta^\gamma)$ for every $\gamma < \alpha(t_i)$. Otherwise, $\alpha(t_i) = 0$. Therefore, for all $t_i \in T_i$, there exists a least ordinal $\alpha(t_i)$ such that $t_i \in T_i\left(\beta^{\alpha(t_i)}\right) \setminus T_i(\beta^\gamma)$ for every $\gamma < \alpha(t_i)$. Lemma 6 implies that for all $t_i \in T_i$, there exists a least ordinal $\alpha(t_i)$ such that $t_i \in T_i^*\left(\beta^{\alpha(t_i)}\right) \setminus T_i^*(\beta^\gamma)$ for every $\gamma < \alpha(t_i)$. Since $t_i \in T_i^*\left(\beta^{\alpha(t_i)}\right)$, Definition 11 implies that there exists $\bar{y}_i \in Y_i^f$ satisfying (8). Let us denote $\bar{y}_i$ by $\bar{y}_i^{\beta^{\alpha(t_i)}}$. Define

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the allocation $\bar{y}_i^{\beta^{\bar{\alpha}(i)}}$ as follows:

$$\bar{y}_i^{\beta^{\bar{\alpha}(i)}} = \sum_{l \geq 0} \frac{1}{2^l} \bar{y}_i^{\beta^{\alpha(t_i^l)}} \tag{11}$$

where $l \geq 0$ is an enumeration of T_i. Since $\bar{y}_i^{\beta^{\bar{\alpha}(i)}} \in Y_{i,s}^f$, we can choose an $\varepsilon > 0$ sufficiently small such that the mapping $\mathfrak{y}_i^{\beta^{\bar{\alpha}(i)}} : T_{-i} \to \Delta(A)$ defined by

$$\mathfrak{y}_i^{\beta^{\bar{\alpha}(i)}}(t_{-i}) = (1-\varepsilon)\, \bar{y}_i^{\beta^{\bar{\alpha}(i)}}(t_{-i}) + \varepsilon \hat{y} \tag{12}$$

is such that $\mathfrak{y}_i^{\beta^{\bar{\alpha}(i)}} \in Y_{i,s}^f$.

Before stating our mechanism, we need the following lemma.

Lemma 11. For all $i \in \mathcal{I}$ and all $t_i \in T_i$, if $t_i \in T_i^*\left(\beta^{\alpha(t_i)}\right) \backslash T_i^*\left(\beta^{\gamma}\right)$ for every $\gamma < \alpha(t_i)$, then for all $\psi_i \in \Delta_{t_i}^{\beta^{\alpha}}(\Theta \times \hat{T}_{-i})$, with $\alpha \geq \alpha(t_i)$, there exists $y_i' \in Y_i^f$ such that

$$\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \psi_i(\theta,\hat{t}_{-i}) u_i(y_i'(\hat{t}_{-i}),\theta) > \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \psi_i(\theta,\hat{t}_{-i}) u_i(\mathfrak{y}_i^{\beta^{\bar{\alpha}(i)}}(\hat{t}_{-i}),\theta). \tag{13}$$

Proof. Fix any $i \in \mathcal{I}$ and any $t_i \in T_i$. Suppose that $t_i \in T_i^*\left(\beta^{\alpha(t_i)}\right) \backslash T_i^*\left(\beta^{\gamma}\right)$ for every $\gamma < \alpha(t_i)$. Definition 11 implies that for all $\psi_i \in \Delta_{t_i}^{\beta^{\alpha(t_i)}}(\Theta \times \hat{T}_{-i})$, there exists $y_i \in Y_i^f$ such that

$$\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \psi_i(\theta,\hat{t}_{-i}) u_i(y_i(\hat{t}_{-i}),\theta) > \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \psi_i(\theta,\hat{t}_{-i}) u_i(\bar{y}_i^{\beta^{\alpha(t_i)}}(\hat{t}_{-i}),\theta).$$

Since $\bar{y}_i^{\beta^{\alpha(t_i)}} \in$Supp$(\bar{y}_i^{\beta^{\bar{\alpha}(i)}})$, we can see that there exists $y_i' \in Y_i^f$ such that the inequality in the statement holds for all $\psi_i \in \Delta_{t_i}^{\beta^{\alpha(t_i)}}(\Theta \times \hat{T}_{-i})$. Since Lemma 3 implies that $\beta^{\alpha} \subseteq \beta^{\alpha(t_i)}$ for all $\alpha \in \Omega$ such that $\alpha(t_i) \geq \alpha$, we can see that the inequality in the statement holds for all $\psi_i \in \Delta_{t_i}^{\beta^{\alpha}}(\Theta \times \hat{T}_{-i})$ with $\alpha(t_i) \geq \alpha$. □

Let us now define the mechanism $\mathcal{M}$. For all $i \in \mathcal{I}$, let

$$M_i = M_i^1 \times M_i^2 \times M_i^3 \times M_i^4,$$

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where

$$M_i^1 = T_i,\ M_i^2 = \mathbb{N},\ M_i^3 = Y_i^* \text{ and } M_i^4 = \Delta^*(A),$$

where $\mathbb{N}$ is the set of natural numbers, Y_i^* is a countable, dense subset of Y_i^f, and $\Delta^*(A)$ is a countable, dense subset of $\Delta(A)$. For all $m \in M$, let $g : M \to \Delta(A)$ be defined as follows.

Rule 1: If $m_i^2 = 1$ for all $i \in \mathcal{I}$, then $g(m) = f(m^1)$.

Rule 2: For all $i \in \mathcal{I}$, if $m_j^2 = 1$ for all $j \in \mathcal{I} \setminus \{i\}$ and $m_i^2 > 1$, then

$$g(m) = m_i^3\left(m_{-i}^1\right)\left(1 - \frac{1}{1+m_i^2}\right) \oplus \mathfrak{y}_i^{\beta^{\bar{\alpha}(i)}}\left(m_{-i}^1\right)\left(\frac{1}{1+m_i^2}\right), \tag{14}$$

where $\mathfrak{y}_i^{\beta^{\bar{\alpha}(i)}} \in Y_{i,s}^f$ is defined in (12).

Rule 3: Otherwise, for each $i \in \mathcal{I}$, m_i^4 is picked with probability $\frac{1}{I}\left(1 - \frac{1}{1+m_i^2}\right)$ and $\hat{y}_i$ is picked with probability $\frac{1}{I}\left(\frac{1}{1+m_i^2}\right)$; that is,

$$g(m) = \frac{1}{I}\left[m_i^4\left(1 - \frac{1}{1+m_i^2}\right) \oplus \hat{y}_i\left(\frac{1}{1+m_i^2}\right)\right], \tag{15}$$

where $\hat{y}_i$ is specified by Lemma 10.

Before we start with the proof, let us make the following remark.

Remark 3. The implementing mechanisms devised by Bergemann and Morris (2011), Bergemann and Morris (2008b), and Oury and Tercieux (2012) are uncountable mechanisms. Müller (2020) shows that restricting attention to countable mechanisms for robust implementation is without loss of generality. In an interim setup, Kunimoto et al. (2020) devise an implementable countable mechanism under a weakening of Assumption 1 (see Theorem 8.1, p. 45, of Kunimoto et al. (2020)).

Suppose that f satisfies IIM on $\mathcal{T}$. In what follows, we prove that $\mathcal{M}$ ICR-implements f on $\mathcal{T}$ and that $\mathcal{M}$ satisfies the EBRP. The following lemmata will help us to complete the proof.

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Lemma 12. $\overline{BNE}\left(U\left(\mathcal{M},\mathcal{T}\right)\right)\neq\emptyset$.

Proof. For all $i\in\mathcal{I}$, let $\sigma_i:T_i\to M_i$ be defined by $\sigma_i\left(t_i\right)=\left(t_i,1,\cdot,\cdot\right)$. For all $i\in\mathcal{I}$ and all $t_i\in T_i$, let $\pi_i\left(t_i\right)\in\Delta\left(\Theta\times T_i\times M_{-i}\right)$ be defined by

$$\pi_i\left(t_i\right)\left[\theta,t_i,m_{-i}\right]=\kappa\left(t_i\right)\left[\theta,t_{-i}\right]\delta_{\sigma_{-i}\left(t_{-i}\right)}\left[m_{-i}\right],$$

where $\delta_{\sigma_{-i}(t_{-i})}$ is the dirac measure on $\left\{\sigma_{-i}\left(t_{-i}\right)\right\}$. By construction, for all $t_i\in T_i$ and all $\left(\theta,t_{-i},m_{-i}\right)\in\Theta\times T_i\times M_{-i}$, $\pi_i\left(t_i\right)\left[\theta,t_i,m_{-i}\right]>0\implies m_{-i}=\sigma_{-i}\left(t_{-i}\right)$. Moreover, by construction and *Rule 1*, for all $i\in\mathcal{I}$ and all $t_i\in T_i$,

$$\begin{aligned}&\sum_{\left(\theta,m_{-i}\right)\in\Theta\times M_{-i}}\operatorname{marg}_{\Theta\times M_{-i}}\pi_i\left(t_i\right)\left[\theta,m_{-i}\right]u_i\left(g\left(\sigma_i\left(t_i\right),m_{-i}\right),\theta\right)\\=&\sum_{\left(\theta,t_{-i}\right)\in\Theta\times T_{-i}}\kappa\left(t_i\right)\left[\theta,t_{-i}\right]u_i\left(f\left(t_i,t_{-i}\right),\theta\right).\end{aligned}$$

Finally, by definition of g and the fact that f is incentive compatible on $\mathcal{T}$ (Lemma 18), it follows that for all $i\in\mathcal{I}$ and all $t_i\in T_i$, $\text{Supp}(\sigma_i\left(t_i\right))\subseteq BR_i\left(\operatorname{marg}_{\Theta\times M_{-i}}\pi_i(t_i,\sigma_{-i})|\mathcal{M}\right)$, and so $\sigma\in\overline{BNE}\left(U\left(\mathcal{M},\mathcal{T}\right)\right)$. $\square$

Before proving the following lemma, let us introduce the following definitions. For all $\beta\in\mathcal{B}$ and all $i\in\mathcal{I}$, define $\Sigma_i^{\beta_i}:T_i\to 2^{M_i}\backslash\left\{\emptyset\right\}$ by

$$\Sigma_i^{\beta_i}\left(t_i\right)=\left\{m_i\in M_i|m_i^1\in\beta_i\left(t_i\right)\right\},\tag{16}$$

and define $\tilde{\Sigma}_i^{\beta_i}:T_i\to 2^{M_i}\backslash\left\{\emptyset\right\}$ by

$$\tilde{\Sigma}_i^{\beta_i}\left(t_i\right)=\left\{m_i\in\Sigma_i^{\beta_i}\left(t_i\right)|m_i^2=1\right\}.\tag{17}$$

It can be checked that $\Sigma^{\beta},\tilde{\Sigma}^{\beta}\in\mathfrak{S}^{\mathcal{M},\mathcal{T}}$.

Lemma 13. For all $(\alpha,i,t_i)\in\Omega\times\mathcal{I}\times T_i(\beta^{\alpha})$ and all $\pi_i(t_i)\in\Delta_{t_i}^{\Sigma_{-i}^{\beta_{-i}^{\alpha}}}\left(\Theta\times T_{-i}\times M_{-i}\right)$, if

$$m_i\in BR_i\left(\operatorname{marg}_{\Theta\times M_{-i}}\pi_i(t_i)|\mathcal{M}\right)\tag{18}$$

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then $m_i^2 = 1$, $\pi_i(t_i) \in \Delta_{t_i}^{\tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}}(\Theta \times T_{-i} \times M_{-i})$ and $m_i^1 \in \beta_i^{\alpha+1}(t_i)$.

Proof. Fix any $(\alpha, i, t_i) \in \Omega \times \mathcal{I} \times T_i(\beta^\alpha)$. Suppose that $\pi_i(t_i) \in \Delta_{t_i}^{\Sigma_{-i}^{\beta_{-i}^{\alpha}}}(\Theta \times T_{-i} \times M_{-i})$ and that $m_i \in BR_i\left(\text{marg}_{\Theta \times M_{-i}} \pi_i(t_i) | \mathcal{M}\right)$. Let us first show that $m_i^2 = 1$. Assume, to the contrary, that $m_i^2 > 1$. Let us proceed according to whether *Rule 2* applies or *Rule 3* applies. To this end, let us note that $\pi_i(t_i) \in \Delta_{t_i}^{\Sigma_{-i}^{\beta_{-i}^{\alpha}}}(\Theta \times T_{-i} \times M_{-i})$ can be decomposed as follows:

$$\underbrace{\sum_{\theta \in \Theta} \sum_{t_{-i} \in T_{-i}} \sum_{m_{-i} \in \tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}(t_{-i})} \pi_i(t_i)[\theta, t_{-i}, m_{-i}]}_{Prob[Rule2]} + \underbrace{\sum_{\theta \in \Theta} \sum_{t_{-i} \in T_{-i}} \sum_{m_{-i} \in M_{-i} \setminus \tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}(t_{-i})} \pi_i(t_i)[\theta, t_{-i}, m_{-i}]}_{Prob[Rule3]} = 1. \tag{19}$$

For all $i \in \mathcal{I}$ and all $t_i \in T_{-i}$, define $\nu_i(t_i) \in \Delta(\Theta \times T_{-i} \times M_{-i}^1)$ by

$$\nu_i(t_i)[\theta, t_{-i}, m_{-i}^1] = \frac{\sum_{m_{-i} \in \tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}(t_{-i})[m_{-i}^1]} \pi_i(t_i)[\theta, t_{-i}, m_{-i}]}{Prob[Rule2]}. \tag{20}$$

Since $\pi_i(t_i) \in \Delta_{t_i}^{\Sigma_{-i}^{\beta_{-i}^{\alpha}}}(\Theta \times T_{-i} \times M_{-i})$, it follows that $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^{\alpha}}(\Theta \times T_{-i}^1 \times M_{-i}^1)$. Let $\psi_i = marg_{\Theta \times M_{-i}^1} \nu_i(t_i)$. Since $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^{\alpha}}(\Theta \times T_{-i}^1 \times M_{-i}^1)$, it holds that

$$\psi_i \in \Delta^{\beta_{-i}^{\alpha}, t_i}(\Theta \times M_{-i}^1). \tag{21}$$

Next, let $\phi_i(\theta) \in \Delta(\Theta)$ be defined by

$$\phi_i(\theta) = \frac{\sum_{t_{-i} \in T_{-i}} \sum_{m_{-i} \in M_{-i} \setminus \tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}(t_{-i})} \pi_i(t_i)[\theta, t_{-i}, m_{-i}]}{Prob[Rule3]}. \tag{22}$$

The utility of m_i under the beliefs $marg_{\Theta \times M_{-i}} \pi_i$, which is denoted by $U_i(m_i, marg_{\Theta \times M_{-i}} \pi_i(t_i))$, is given by

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$$U_i(m_i, marg_{\Theta\times M_{-i}}\pi_i(t_i)) = \alpha \sum_{(\theta,t_{-i})\in\Theta\times\hat{T}_{-i}} \psi_i(\theta,t_{-i})u_i\Big[\big(1-\frac{1}{m_i^2+1})m_i^3(t_{-i}) \oplus \frac{1}{m_i^2+1}\mathfrak{y}_i^{\beta^{\bar{\alpha}(i)}}(t_{-i})\big),\theta\Big]$$
$$+ (1-\alpha)\sum_{\theta\in\Theta}\phi_i(\theta)u_i\Big[\big(1-\frac{1}{m_i^2+1})m_i^4 \oplus \frac{1}{m_i^2+1}\hat{y}_i\big),\theta\Big] \tag{23}$$

where $\alpha = Prob[Rule2]$.

Since $\psi_i \in \Delta^{\beta^{\alpha}_{-i},t_i}(\Theta\times\hat{T}_{-i})$ and $t_i \in T_i(\beta^{\alpha})$ and since $\alpha \geq \alpha(t_i)$, Lemma 11 implies that there exists $y_i' \in Y_i^f$ such that

$$\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \psi_i(\theta,\hat{t}_{-i})u_i(y_i'(\hat{t}_{-i}),\theta) > \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \psi_i(\theta,\hat{t}_{-i})u_i(\mathfrak{y}_i^{\beta^{\bar{\alpha}(i)}}(\hat{t}_{-i}),\theta). \tag{24}$$

Furthermore, Lemma 10 implies that there exists $y_i \in \Delta(A)$ such that

$$\sum_{\theta\in\Theta}\phi_i(\theta)u_i(y_i,\theta) > \sum_{\theta\in\Theta}\phi_i(\theta)u_i(\hat{y}_i,\theta). \tag{25}$$

Since $m_i \in BR_i(marg_{\Theta\times M_{-i}}\pi_i(t_i)|\mathcal{M})$, it follows that

$$\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \psi_i(\theta,\hat{t}_{-i})u_i(m_i^3(\hat{t}_{-i}),\theta) \geq \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \psi_i(\theta,\hat{t}_{-i})u_i(y_i'(\hat{t}_{-i}),\theta) \tag{26}$$

and that

$$\sum_{\theta\in\Theta}\phi_i(\theta)u_i(m_i^4,\theta) \geq \sum_{\theta\in\Theta}\phi_i(\theta)u_i(y_i,\theta). \tag{27}$$

Inequalities in (24)-(27) imply that $U_i(m_i, marg_{\Theta\times M_{-i}}\pi_i(t_i))$ is strictly increasing in m_i^2, which is a contradiction. Thus, $m_i^2 = 1$.

Next, let us show that $\pi_i(t_i) \in \Delta_{t_i}^{\tilde{\Sigma}_{-i}^{\beta^{\alpha}_{-i}}}(\Theta\times T_{-i}\times M_{-i})$. Assume, to the contrary, that $\pi_i(t_i) \notin \Delta_{t_i}^{\tilde{\Sigma}_{-i}^{\beta^{\alpha}_{-i}}}(\Theta\times T_{-i}\times M_{-i})$. Then, since $m_i^2 = 1$, either *Rule 2* applies where $m_j^2 > 1$ for some $j \in \mathcal{I}\backslash\{i\}$ or *Rule 3* applies. In what follows, we focus only on the

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case that *Rule 2* applies.[23]

By the definition of g, for all $(\theta, m_{-i}) \in$Supp$(\underset{\Theta\times M_{-i}}{marg}\pi_i(t_i))$, it holds that

$$g(m_i, m_{-i}) = (1 - \frac{1}{m_j^2+1})m_j^3(m_{-j}^1) + \frac{1}{m_j^2+1}\mathfrak{y}_j^{\beta^{\bar{\alpha}(j)}}(m_{-j}^1), \tag{28}$$

where, for $\varepsilon > 0$ sufficiently small,

$$\mathfrak{y}_j^{\beta^{\bar{\alpha}(j)}}(t_{-j}) = (1-\varepsilon)\,\bar{y}_j^{\beta^{\bar{\alpha}(j)}}(t_{-j}) + \varepsilon\hat{y}. \tag{29}$$

To show that player i can gain by triggering *Rule 3*, we need to define a lottery $\hat{m}_i^4 \in \Delta^*(A) = M_i^4$ that can be used by player i. To this end, we first define the allocation h over M as follows: For all (m_i, m_{-i}) such that $(\theta, m_{-i}) \in$Supp$(\underset{\Theta\times M_{-i}}{marg}\pi_i(t_i))$,

$$h(m_i, m_{-i}) = (1 - \frac{1}{m_j^2+1})m_j^3(m_{-j}^1) + \frac{1}{m_j^2+1}\tilde{\mathfrak{y}}_j^{\beta^{\bar{\alpha}(j)}}(m_{-j}^1) \tag{30}$$

where $\tilde{\mathfrak{y}}_j^{\beta^{\bar{\alpha}(j)}}(t_{-j})\,(1-\varepsilon)\,\bar{y}_j^{\beta^{\bar{\alpha}(j)}}(t_{-j}) + \varepsilon[\sum_{j\neq i}\frac{1}{I}\hat{y}_j + \frac{1}{I}y_i]$ and y_i is such that (10) is satisfied. Finally, let us define $\hat{m}_i^4$ by

$$\hat{m}_i^4 = \sum marg_{\Theta\times M_{-i}}\pi_i(t_i)(\theta, m_{-i})h(\cdot, m_{-i}). \tag{31}$$

Since player i's utility is strictly higher under $h(m_i, m_{-i})$ than under $g(m_i, m_{-i})$ for each $(\theta, m_{-i}) \in$Supp$(\underset{\Theta\times M_{-i}}{marg}\pi_i(t_i))$ and since, moreover, player i's utility function is continuous, we can assume without loss of generality that $\hat{m}_i^4 \in \Delta^*(A) = M_i^4$.

Since player i's utility is strictly higher under $h(m_i, m_{-i})$ than under $g(m_i, m_{-i})$, for every $(\theta, m_{-i}) \in$Supp$(\underset{\Theta\times M_{-i}}{marg}\pi_i(t_i))$, player i can change m_i with $m_i' \in M_i$, where its fourth component is $\hat{m}_i^4$ and its second component is $\hat{m}_i^2 > 1$, so that he can trigger *Rule 3*. Since the utility gain of player i is obtained point-wise in the Supp$(\underset{\Theta\times M_{-i}}{marg}\pi_i(t_i))$, we obtain the desired contradiction. Thus, $\pi_i(t_i) \in \Delta_{t_i}^{\tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}}(\Theta \times T_{-i} \times M_{-i})$.

[23]When *Rule 3* applies, we can see, by the arguments provided above, that player i can find a profitable deviation.

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Finally, let us show that $m_i^1 \in \beta_i^{\alpha+1}(t_i)$. Since $\pi_i(t_i) \in \Delta_{t_i}^{\tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}}(\Theta \times T_{-i} \times M_{-i})$, we have that

$$\sum_{t_{-i}\in T_{-i}} \sum_{m_{-i}\in \tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}(t_{-i})} \pi_i(t_i)[\theta, t_{-i}, m_{-i}] = 1.$$

Let $\nu_i(t_i) \in \Delta\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ be defined by

$$\nu_i(t_i)\left[\theta, t_{-i}, m_{-i}^1\right] = \sum_{m_{-i}\in \tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}\left(m_{-i}^1\right)} \pi_i(t_i)[\theta, t_{-i}, m_{-i}]. \tag{32}$$

By definition, we can see that $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^{\alpha}}\left(\Theta \times T_{-i} \times M_{-i}^1\right)$. Since $m_1^2 = 1$, then *Rule 1* applies with probability 1, and so

$$\begin{aligned} &\textstyle\sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\mathrm{marg}_{\Theta\times M_{-i}}\pi_i(t_i)[\theta, m_{-i}]\right) u_i\left(g\left(m_i, m_{-i}\right), \theta\right) = \\ &\textstyle\sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\mathrm{marg}_{\Theta\times M_{-i}}\pi_i(t_i)[\theta, m_{-i}]\right) u_i\left(f\left(m_i^1, m_{-i}^1\right), \theta\right), \end{aligned} \tag{33}$$

and so, by (32),

$$\begin{aligned} &\textstyle\sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\mathrm{marg}_{\Theta\times M_{-i}}\pi_i(t_i)[\theta, m_{-i}]\right) u_i\left(f\left(m_i^1, m_{-i}^1\right), \theta\right) = \\ &\textstyle\sum_{\left(\theta,m_{-i}^1\right)\in\Theta\times M_{-i}^1} \left(\mathrm{marg}_{\Theta\times M_{-i}^1}\nu_i(t_i)\left[\theta, m_{-i}^1\right]\right) u_i\left(f\left(m_i^1, m_{-i}^1\right), \theta\right). \end{aligned}$$

Moreover, since $m_i \in BR_i\left(\mathrm{marg}_{\Theta\times M_{-i}}\pi_i(t_i)|\mathcal{M}\right)$ and since, moreover, player i can never induce *Rule 3*, it follows from the definition of g that

$$\begin{aligned} &\textstyle\sum_{\left(\theta,m_{-i}^1\right)\in\Theta\times M_{-i}^1} \left(\mathrm{marg}_{\Theta\times M_{-i}^1}\nu_i(t_i)\left[\theta, m_{-i}^1\right]\right) u_i\left(f\left(m_i^1, m_{-i}^1\right), \theta\right) \geq \\ &\textstyle\sum_{\left(\theta,m_{-i}^1\right)\in\Theta\times M_{-i}^1} \left(\mathrm{marg}_{\Theta\times M_{-i}^1}\nu_i(t_i)\left[\theta, m_{-i}^1\right]\right) u_i\left(m_i^3\left(m_{-i}^1\right), \theta\right), \end{aligned} \tag{34}$$

for all $m_i^3 \in Y_i^*$. Since Y_i^* is a countable, dense subset of Y_i^f and since u_i is continuous, we have that the inequality in (34) holds for all $m_i^3 \in Y_i^f$. Since $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^{\alpha}}\left(\Theta \times T_{-i} \times M_{-i}^1\right)$ and since, moreover, the inequality in (34) holds for all $m_i^3 \in Y_i^f$, and $m_i^1 \in \beta_i^{\alpha}(t_i)$, it follows from (4) that $m_i^1 \in \beta_i^{\alpha+1}(t_i)$, as we sought. □

Lemma 14. For all $\alpha \in \Omega$ and all $i \in \mathcal{I}$, $S_i^{\alpha,\mathcal{M},\mathcal{T}} \subseteq \Sigma_i^{\beta_i^{\alpha}}$

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Proof. Let us proceed by transfinite induction over Ω. It is clear that $S_i^{\alpha,\mathcal{M},\mathcal{T}} \subseteq \Sigma_i^{\beta_i^\alpha} = M_i$ for all $i \in \mathcal{I}$ if $\alpha = 0$. Fix any $\alpha \in \Omega \setminus \{0\}$. Suppose that for all $\gamma < \alpha$, $S_i^{\gamma,\mathcal{M},\mathcal{T}} \subseteq \Sigma_i^{\beta_i^\gamma}$ for all $i \in \mathcal{I}$. Fix any $i \in \mathcal{I}$. We proceed according to whether α is a successor ordinal or not.

Suppose that α is a limit ordinal. Since $\bigcap_{\gamma<\alpha} S_i^{\gamma,\mathcal{M},\mathcal{T}} = S_i^{\alpha,\mathcal{M},\mathcal{T}}$, by Definition 3, it follows that $S_i^{\alpha,\mathcal{M},\mathcal{T}} \subseteq \bigcap_{\gamma<\alpha} \Sigma_i^{\beta_i^\gamma}$. Fix any $t_i \in T_i$ and any $m_i \in \bigcap_{\gamma<\alpha} \Sigma_i^{\beta_i^\gamma}(t_i)$. Then, $m_i^1 \in \bigcap_{\gamma<\alpha} \beta_i^\gamma(t_i)$. It follows from (5) that $m_i^1 \in \beta_i^\alpha(t_i)$. Since the choice of $t_i \in T_i$ was arbitrary, we have that $\bigcap_{\gamma<\alpha} \Sigma_i^{\beta_i^\gamma} \subseteq \Sigma_i^{\beta_i^\alpha}$. Since $S_i^{\alpha,\mathcal{M},\mathcal{T}} \subseteq \bigcap_{\gamma<\alpha} \Sigma_i^{\beta_i^\gamma}$, we have that $S_i^{\alpha,\mathcal{M},\mathcal{T}} \subseteq \Sigma_i^{\beta_i^\alpha}$.

Suppose that α is a successor ordinal. Fix any $t_i \in T_i$. We proceed according to whether $t_i \in T_i(\beta^{\alpha-1})$ or not. Suppose that $t_i \in T_i(\beta^{\alpha-1})$. Fix any $m_i \in S_i^{\alpha,\mathcal{M},\mathcal{T}}(t_i)$. The inductive hypothesis implies that $S_{-i}^{\alpha-1,\mathcal{M},\mathcal{T}} \subseteq \Sigma_{-i}^{\beta_{-i}^{\alpha-1}}$.

Since $m_i \in S_i^{\alpha,\mathcal{M},\mathcal{T}}(t_i)$, Definition 3 implies that $m_i \in S_i^{\alpha-1,\mathcal{M},\mathcal{T}}$ and that there exists $\pi_i \in \Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i})$ such that $\pi_i(t_i) \in \Delta_{t_i}^{S_{-i}^{\alpha-1,\mathcal{M},\mathcal{T}}}(\Theta \times T_{-i} \times M_{-i})$ and that $m_i \in \mathrm{BR}_i\left(\mathrm{marg}_{\Theta\times M_{-i}}\pi_i(t_i)|\mathcal{M}\right)$. Since $S_{-i}^{\alpha-1,\mathcal{M},\mathcal{T}} \subseteq \Sigma_{-i}^{\beta^{\alpha-1}}$, it follows that

$$\pi_i(t_i) \in \Delta_{t_i}^{\Sigma_{-i}^{\beta^{\alpha-1}}}(\Theta \times T_{-i} \times M_{-i}).$$

Since $t_i \in T_i(\beta^{\alpha-1})$ and since, moreover, $m_i \in BR_i\left(\mathrm{marg}_{\Theta\times M_{-i}}\pi_i(t_i)|\mathcal{M}\right)$ and $\pi_i(t_i) \in \Delta_{t_i}^{\Sigma_{-i}^{\beta_{-i}^{\alpha-1}}}(\Theta \times T_{-i} \times M_{-i})$, Lemma 26 implies that $m_i^2 = 1$ and that $m_i^1 \in \beta_i^\alpha(t_i)$. Thus, $m_i \in \Sigma_i^{\beta_i^\alpha}(t_i)$.

Suppose that $t_i \in T_i^c(\beta^{\alpha-1})$. Lemma 7 implies that $\beta_i^\alpha(t_i) = \bar{\beta}_i(t_i)$. It follows from (16) that $S_i^{\alpha,\mathcal{M},\mathcal{T}}(t_i) \subseteq \Sigma_i^{\beta^\alpha}(t_i)$.

Since the choice of player i and of player i's type t_i were arbitrary, we conclude that for all $i \in \mathcal{I}$, $S_i^{\alpha,\mathcal{M},\mathcal{T}} \subseteq \Sigma_i^{\beta_i^\alpha}$. By the principle of transfinite induction, the statement follows. □

Lemma 15. For all $\alpha \in \Omega$, all $i \in \mathcal{I}$, and all $t_i \in T_i(\beta^\alpha)$, if $m_i \in S_i^{\alpha+1,\mathcal{M},\mathcal{T}}(t_i)$, then $m_i^2 = 1$ and $m_i^1 \in \beta_i^{\alpha+1}(t_i)$.

Proof. Fix $(\alpha, i, t_i) \in \Omega \times \mathcal{I} \times T_i(\beta^\alpha)$. Suppose that $m_i \in S_i^{\alpha+1,\mathcal{M},\mathcal{T}}(t_i)$. Definition 3 implies that there exists $\pi_i(t_i) \in \Delta_{t_i}^{S_{-i}^{\alpha,\mathcal{M},\mathcal{T}}}(\Theta \times T_{-i} \times M_{-i})$ such that $m_i \in$

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$BR_i\left(\text{marg}_{\Theta\times M_{-i}}\pi_i(t_i)|\mathcal{M}\right)$. Lemma 14 implies that

$$S_{-i}^{\alpha,\mathcal{M},\mathcal{T}} \subseteq \Sigma_{-i}^{\beta_{-i}^{\alpha}}, \tag{35}$$

and so $\pi_i(t_i) \in \Delta_{t_i}^{\Sigma_{-i}^{\beta_{-i}^{\alpha}}}(\Theta \times T_{-i} \times M_{-i})$. Lemma 26 implies that $m_i^2 = 1$ and that $m_i^1 \in \beta_i^{\alpha+1}(t_i)$.

Since the choice of $(\alpha, i, t_i) \in \Omega \times \mathcal{I} \times T_i(\beta^\alpha)$ was arbitrary, the proof is complete.

□

Let us show that $\mathcal{M}$ ICR-implements f on $\mathcal{T}$. Lemma 12 implies that for all $i \in \mathcal{I}$ and $t_i \in T_i$, $S_i^{\mathcal{M},\mathcal{T}}(t_i) \neq \emptyset$. Thus, part (i) of Definition 4 is satisfied. Recall that Lemma 3 implies that there exists an α such that $\beta^\alpha = \beta^{\alpha+1} = \beta^*$. Recall that by Lemma 8, we are under the assumption that $T(\beta^*) = T$. Thus, $T(\beta^\alpha) = T$. Fix any $t \in T$ and any $m \in S^{\mathcal{M},\mathcal{T}}(t)$. Since $S^{\mathcal{M},\mathcal{T}}(t) \subseteq S^{\alpha+1,\mathcal{M},\mathcal{T}}(t)$, then $m \in S^{\alpha+1,\mathcal{M},\mathcal{T}}(t)$. Lemma 15 implies that $m_i^2 = 1$ and $m_i^1 \in \beta_i^{\alpha+1}(t_i) = \beta_i^*(t_i)$ for all $(i, t_i) \in \mathcal{I} \times T_i$. *Rule 1* implies that $g(m) = f(m^1)$. Since f satisfies IIM on $\mathcal{T}$, it follows that β^* is an acceptable deception on $\mathcal{T}$ for f. This implies that $f(m^1) = f(t)$. Since the choice of $(t, m) \in T \times S^{\mathcal{M},\mathcal{T}}(t)$ was arbitrary, we conclude that part (ii) of Definition 4 is satisfied. Thus, f is ICR-implementable on $\mathcal{T}$. Finally, in light of Remark 1, Lemma 12 implies that $\mathcal{M}$ also implements f in Bayes–Nash equilibria.

VI. Concluding remarks

This paper shows that IRM is necessary and sufficient for interim rationalizable (and Bayes–Nash) implementation. Moreover, it sheds light on the role played by Assumption 1 of Oury and Tercieux (2012). These contributions can serve as a point of departure for answering important research questions. We discuss some of them below.

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Relaxing the double implementation requirement

This paper shows that IRM is necessary and sufficient for implementation in interim rationalizable strategies by a mechanism having a pure strategy Bayes-Nash equilibrium. We achieved this result by suitably modifying the mechanism of Oury and Tercieux (2012), which is devised under Assumption 1. This modification is based on an iterative version of IRM, called IIM.

By adapting the arguments used to develop the iterative version of IRM, an iterative version of the weak IRM can be formulated. This iterative condition can then be shown to be sufficient for implementation in interim rationalizable strategies. This result can be obtained by modifying the mechanism devised by Kunimoto et al. (2020) in a way that we have modified the mechanism devised by Oury and Tercieux (2012). Details are available in Appendix B.

Robust Implementation

An SCF is robustly implementable if every equilibrium on every type space achieves outcomes consistent with it. A seminal paper on robust implementation in general environments is Bergemann and Morris (2011). They show that the conditions for robust implementation can be derived as an implication of rationalizable implementation. Moreover, Jain et al. (2022b) provide a notion of rationalizable implementation that is equivalent to robust implementation. Thus, necessary and sufficient conditions for robust implementation can be provided by adapting the techniques developed in this paper for rationalizable (and Bayes-Nash) implementation on a fixed, arbitrary type space. We conjecture that robust monotonicity, due to Bergemann and Morris (2011), is necessary and sufficient for robust implementation. We are pursuing this conjecture in one of our ongoing works (Jain and Lombardi (2022)).

Implementation via extensive form games

Müller (2020) introduces and studies a strong form of robust implementation in dynamic mechanisms that is both belief- and belief-revision-free. Specifically, he studies

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full implementation problems in weak Perfect Bayesian equilibrium across all type spaces. He presents a necessary condition for implementation, named dynamic robust monotonicity, which is weaker than the robust monotonicity condition due to Bergemann and Morris (2011). Moreover, he shows that under a conditional no total indifference condition, ex-post incentive compatibility and dynamic robust monotonicity characterize robust implementation in weak Perfect Bayesian equilibrium by general dynamic mechanisms. It is important to note that Müller (2020)'s work can be adapted to study implementation problems by a general dynamic mechanism on a fixed, arbitrary type space. Specifically, a dynamic IRM condition that is weaker than IRM can be formulated as a necessary condition for implementation. Furthermore, we believe that our work on interim rationalizable (and Bayes–Nash) implementation can be adapted to identify the class of implementable functions by extensive form games.

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Appendices

A. Proof of Theorem 1: Part (i) implies part (ii)

By adopting the arguments of Bergemann and Morris (2011) to the interim setup, Bergemann and Morris (2008b) (Proposition 4) shows in a payoff type space that IRM is necessary for ICR-implementation by a mechanism satisfying BRP. Theorem 3 of Oury and Tercieux (2012) provides arguments to show that IRM is a necessary condition for implementation both in interim rationalizable strategies and Nash equilibrium strategies in a general type space. We present it for the sake of completeness.

Let $\mathcal{T}$ be any model. Let $f : T \to \Delta(A)$ be any SCF. Assume that $\mathcal{M}$ satisfies the EBRP and that $\mathcal{M}$ ICR-implements f. Lemma 2 implies that there exists a pure strategy $\sigma \in BNE(U(\mathcal{M}, \mathcal{T}))$. This implies that for all $i \in \mathcal{I}$ and all $t_i \in T_i$,

$$\begin{array}{ll} \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(t_i)[\theta, t_{-i}]\, u_i(g(\sigma(t)), \theta) & \geq \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(t_i)[\theta, t_{-i}]\, u_i(g((m_i, \sigma_{-i}(t_{-i}))), \theta) & \end{array}$$

for all $m_i \in M_i$. Since $\mathcal{M}$ ICR-implements f, it follows that for all $i \in \mathcal{I}$ and all $t_i \in T_i$,

$$\begin{array}{ll} \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(t_i)[\theta, t_{-i}]\, u_i(f(t), \theta) & \geq \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(t_i)[\theta, t_{-i}]\, u_i(g((m_i, \sigma_{-i}(t_{-i}))), \theta) & \end{array} \tag{36}$$

for all $m_i \in M_i$.

Suppose that the deception β is unacceptable. For all $i \in \mathcal{I}$ and all $t_i \in T_i$, $\Sigma_i(t_i) = \{\sigma_i(t'_i) \in M_i | t'_i \in \beta_i(t_i)\}$. Then, Σ_i is a correspondence from T_i to $2^{M_i} \setminus \{\emptyset\}$, and so $\Sigma_i \in \mathfrak{S}_i^{\mathcal{M},\mathcal{T}}$. Since $\mathcal{M}$ ICR-implements f, it follows that $\Sigma \in \mathfrak{S}^{\mathcal{M},\mathcal{T}}$ cannot be a best-reply set in $U(\mathcal{M}, \mathcal{T})$. Then, for some $(i, t_i, \sigma(\hat{t}_i)) \in \mathcal{I} \times T_i \times \Sigma_i(t_i)$ and all $\pi_i(t_i) \in \Delta_{t_i}^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i})$, it holds that

$$\sigma_i(\hat{t}_i) \notin BR_i\left(\text{marg}_{\Theta\times M_{-i}}\pi_i(t_i) | \mathcal{M}\right),$$

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and so

$$\begin{array}{lr} \sum_{(\theta,m_{-i})\in\Theta\times M_{-i}}\left(\text{marg}_{\Theta\times M_{-i}}\pi_i\left(t_i\right)[\theta,m_{-i}]\right)\left[u_i\left(g\left(m_i,m_{-i}\right),\theta\right)\right] & > \\ \sum_{(\theta,m_{-i})\in\Theta\times M_{-i}}\left(\text{marg}_{\Theta\times M_{-i}}\pi_i\left(t_i\right)[\theta,m_{-i}]\right)\left[u_i\left(g\left(\sigma_i\left(\hat{t}_i\right),m_{-i}\right),\theta\right)\right] & \end{array} \tag{37}$$

for some $m_i \in M_i$.

For all $i \in \mathcal{I}$ and all $t_i \in T_i$, let $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ be any distribution. For all $i \in \mathcal{I}$, all $t_i \in T_i$, let $\bar{\pi}_i(t_i) \in \Delta(\Theta \times T_{-i} \times M_{-i})$ be defined, for all $(\theta, t_{-i}, m_{-i}) \in \Theta \times T_{-i} \times M_{-i}$, by

$$\bar{\pi}_i(t_i)[\theta, t_{-i}, m_{-i}] = \sum_{\hat{t}_{-i} \in \sigma_{-i}^{-1}(m_{-i})} \nu_i(t_i)\left[\theta, t_{-i}, \hat{t}_{-i}\right],$$

where $\sigma_{-i}^{-1}(m_{-i}) = \Pi_{j \in \mathcal{I}\setminus\{i\}} \sigma_j^{-1}(m_j)$ and $\sigma_j^{-1}(m_j) = \{t_j \in T_j | m_j = \sigma_j(t_j)\}$. Since $\nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$, we have that $\text{marg}_{\Theta\times T_{-i}}\nu_i(t_i) = \kappa(t_i)$. Moreover, by construction, $\text{marg}_{\Theta\times T_{-i}}\nu_i(t_i) = \text{marg}_{\Theta\times T_{-i}}\bar{\pi}_i(t_i)$.[24] Moreover, since $\nu_i(t_i)$ belongs to $\Delta_{t_i}^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$, it also follows that for all $(\theta, t_{-i}, m_{-i}) \in \Theta \times T_{-i} \times M_{-i}$, $\bar{\pi}_i(t_i)[\theta, t_{-i}, m_{-i}] > 0 \implies m_{-i} \in \Sigma_{-i}(t_{-i})$. Thus, we have that $\bar{\pi}_i(t_i) \in \Delta_{t_i}^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i})$ for all $i \in \mathcal{I}$ and all $t_i \in T_i$. Moreover, by construction, we

[24]Observe that for all $(\theta, t_{-i}) \in \Theta \times T_{-i}$,

$$\begin{aligned} \text{marg}_{\Theta\times T_{-i}}\bar{\pi}_i(t_i)[\theta, t_{-i}] &= \sum_{m_{-i}\in M_{-i}} \bar{\pi}_i(t_i)[\theta, t_{-i}, m_{-i}] \\ &= \sum_{m_{-i}\in M_{-i}} \left(\sum_{\hat{t}_{-i}\in\sigma_{-i}^{-1}(m_{-i})} \nu_i(t_i)\left[\theta, t_{-i}, \hat{t}_{-i}\right] \right) \\ &= \sum_{\hat{t}_{-i}\in\hat{T}_{-i}} \nu_i(t_i)\left[\theta, t_{-i}, \hat{t}_{-i}\right] \\ &= \text{marg}_{\Theta\times T_{-i}}\nu_i(t_i)[\theta, t_{-i}]. \end{aligned}$$

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also have that for all $i \in \mathcal{I}$ and all $m_i \in M_i$,[25]

$$\sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\text{marg}_{\Theta\times M_{-i}}\bar{\pi}_i(t_i)[\theta,m_{-i}]\right) u_i(g(m_i,m_{-i}),\theta) = \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \left(\text{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(g\left(m_i,\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right). \tag{38}$$

Since $\bar{\pi}_i(t_i) \in \Delta_{t_i}^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i})$ for all $i \in \mathcal{I}$ and all $t_i \in T_i$, from (37) and (38), we have that for some $\left(i, t_i, \sigma\left(\hat{t}_i\right)\right) \in \mathcal{I} \times T_i \times \Sigma_i(t_i)$,

$$\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \left(\text{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(g\left(m_i,\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right) > \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \left(\text{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(g\left(\sigma_i\left(\hat{t}_i\right),\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right). \tag{39}$$

Define $y_i(\cdot) = g(m_i, \sigma_{-i}(\cdot))$. (36) implies that $y_i \in Y_i^f$. Thus, f satisfies IRM on $\mathcal{T}$.

B. A Full characterization of ICR-Implementable Functions

In this Appendix, we show that weak IRM, due to Kunimoto et al. (2020), fully characterizes ICR-implementation. The proof of sufficiency is an adaptation of our proof of Theorem 1.

[25] To see it, observe that

$$\begin{aligned}
& \sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\text{marg}_{\Theta\times M_{-i}}\bar{\pi}_i(t_i)[\theta,m_{-i}]\right) u_i(g(m_i,m_{-i}),\theta) \\
= & \sum_{(\theta,t_{-i},m_{-i})\in\Theta\times T_{-i}\times M_{-i}} \bar{\pi}_i(t_i)[\theta,t_{-i},m_{-i}]\, u_i(g(m_i,m_{-i}),\theta) \\
= & \sum_{(\theta,t_{-i},m_{-i})\in\Theta\times T_{-i}\times M_{-i}} \left(\sum_{\hat{t}_{-i}\in\sigma_{-i}^{-1}(m_{-i})} \nu_i(t_i)\left[\theta,t_{-i},\hat{t}_{-i}\right] u_i\left(g\left(m_i,\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right)\right) \\
= & \sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\sum_{\hat{t}_{-i}\in\sigma_{-i}^{-1}(m_{-i})} \left(\text{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(g\left(m_i,\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right)\right) \\
= & \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \left(\text{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(g\left(m_i,\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right).
\end{aligned}$$

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For all $i \in \mathcal{I}$, let $\bar{Y}_i^f$ be the set of mappings from T to $\Delta(A)$ defined as follows.

$$\bar{Y}_i^f = \left\{ y : T \to \Delta(A) \;\middle|\; \begin{array}{l} \text{For all } \tilde{t}_i \in T_i, \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(\tilde{t}_i)[\theta,t_{-i}]\, u_i\left(f\left(\tilde{t}_i,t_{-i}\right),\theta\right) \geq \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(\tilde{t}_i)[\theta,t_{-i}]\, u_i\left(y\left(\tilde{t}_i,t_{-i}\right),\theta\right). \end{array} \right\} \quad (40)$$

Note that $\bar{Y}_i^f$ is a metrizable separable space. We write $\bar{Y}^f$ for $\Pi_{i\in\mathcal{I}}\bar{Y}_i^f$. For all $i \in \mathcal{I}$, let $Y_{i,s}^f$ be the set of all mappings in $\bar{Y}_i^f$ satisfying the inequality in (40) strictly for all $\tilde{t}_i \in T_i$.[26] Similarly, we write $\overline{Y}_s^f$ for $\Pi_{i\in\mathcal{I}}\overline{Y}_{i,s}^f$.

The following condition is due to Kunimoto et al. (2020). To introduce it, we need additional notation. For all $i \in \mathcal{I}$ and all $t_i \in T_i$, let $\Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}\right)$ be defined by

$$\Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}\right) = \left\{\nu_i \in \Delta\left(\Theta \times T_{-i} \times \hat{T}\right) | \text{marg}_{\Theta\times T_{-i}}\nu_i = \kappa(t_i)\right\},$$

and, moreover, for all $\beta \in \mathcal{B}$, let $\Delta_{t_i}^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}\right)$ be defined by

$$\Delta_{t_i}^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}\right) = \left\{ \nu_i \;\middle|\; \begin{array}{l} \nu_i \in \Delta\left(\Theta \times T_{-i} \times \hat{T}\right) \text{ and} \\ \nu_i\left[\theta, t_{-i}, \hat{t}\right] > 0 \implies \hat{t}_{-i} \in \beta_{-i}(t_{-i}) \end{array} \right\}.$$

For the sake of brevity, we write $\Delta_{t_i}^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}\right)$ for $\Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}\right) \cap \Delta^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}\right)$.

Definition 12. An SCF $f : T \to \Delta(A)$ satisfies *weak IRM* (w-IRM, henceforth) on $\mathcal{T}$ if for all unacceptable deception profile $\beta \in \mathcal{B}$ on $\mathcal{T}$ for f, there exists $(i, t_i, t_i') \in$

[26] Formally, for all $i \in \mathcal{I}$,

$$\bar{Y}_{i,s}^f = \left\{ y : T_{-i} \to \Delta(A) \;\middle|\; \begin{array}{l} \text{For all } \tilde{t}_i \in T_i, \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(\tilde{t}_i)[\theta,t_{-i}]\, u_i\left(f\left(\tilde{t}_i,t_{-i}\right),\theta\right) > \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(\tilde{t}_i)[\theta,t_{-i}]\, u_i\left(y(t_{-i}),\theta\right). \end{array} \right\}$$

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$\mathcal{I} \times T_i \times \beta_i(t_i)$ such that for all $\psi_i(t_i) \in \Delta_{t_i}^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}\right)$, there exists $y_i^* \in \bar{Y}_i^f$ such that

$$\begin{aligned} &\textstyle\sum_{(\theta,\hat{t}) \in \Theta \times \hat{T}} \left(\text{marg}_{\Theta \times \hat{T}} \psi_i(t_i)\left[\theta, \hat{t}\right]\right) u_i\left(y_i^*\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right) \quad > \\ &\textstyle\sum_{(\theta,\hat{t}) \in \Theta \times \hat{T}} \left(\text{marg}_{\Theta \times \hat{T}} \psi_i(t_i)\left[\theta, \hat{t}\right]\right) u_i\left(f\left(t_i', \hat{t}_{-i}\right), \theta\right). \end{aligned} \tag{41}$$

A condition, which is equivalent to w-IRM, can be expressed in terms of the limit point of an iterative net of deception profiles.

The iterative net, denoted by $(\beta^\alpha)_{\alpha \in \Omega}$, is defined as follows. For all $i \in \mathcal{I}$ and all $t_i \in T_i$, let

$$\beta_i^0(t_i) = \bar{\beta}_i(t_i) = T_i,$$

and, for all ordinal numbers $\alpha \in \Omega$, define $\beta_i^\alpha(t_i)$ as follows:

- If α is a successor ordinal, then

$$\beta_i^\alpha(t_i) = \left\{ \hat{t}_i \;\middle|\; \begin{array}{l} \hat{t}_i \in \beta_i^{\alpha-1}(t_i) \text{ and there exists} \\ \nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}\right) \text{ such} \\ \text{that } \nu_i(t_i) \in \Delta^{\beta_{-i}^{\alpha-1}}\left(\Theta \times T_{-i} \times \hat{T}\right) \text{ and} \\ \sum_{(\theta,\hat{t}) \in \Theta \times \hat{T}} \left(\text{marg}_{\Theta \times \hat{T}} \nu_i(t_i)\left[\theta, \hat{t}\right]\right) u_i\left(f\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right) \geq \\ \sum_{(\theta,\hat{t}) \in \Theta \times \hat{T}} \left(\text{marg}_{\Theta \times \hat{T}} \nu_i(t_i)\left[\theta, \hat{t}\right]\right) u_i\left(y_i\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right), \\ \text{for all } y_i \in \bar{Y}_i^f. \end{array} \right\} \tag{42}$$

- If α is a limit ordinal, then

$$\beta_i^\alpha(t_i) = \bigcap_{\gamma < \alpha} \beta_i^\gamma(t_i). \tag{43}$$

Observe that $t_i \in \beta_i^\alpha(t_i)$ for all $i \in \mathcal{I}$, all $t_i \in T_i$ and all $\alpha \in \Omega$. A net $(\beta^\alpha)_{\alpha \in \Omega}$ is monotonic decreasing if $\beta^{\alpha+1} \subseteq \beta^\alpha$ for all $\alpha \in \Omega$. If the limit of $(\beta^\alpha)_{\alpha \in \Omega}$ exists, we denote it by β^*; that is, $\lim_{\alpha \in \Omega} \beta^\alpha \to \beta^*$.

Lemma 16. Let $\mathcal{T}$ be any model. $(\beta^\alpha)_{\alpha \in \Omega}$ is a monotonic decreasing net such that $\lim_{\alpha \in \Omega} \beta^\alpha \to \beta^* \in \mathcal{B}$. Moreover, there exists an ordinal $\alpha \in \Omega$ such that

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$\beta^{\alpha} = \beta^{\alpha+1} = \beta^{*}$.

Proof. Since the proof is an easy adaptation of the proof of Lemma 3, we omit it here. □

Our condition can be stated as follows.

Definition 13. $f : T \to \Delta(A)$ satisfies *weak IIM* (w-IIM, henceforth) on $\mathcal{T}$ if β^* is an acceptable deception on $\mathcal{T}$ for f.

The following result shows that w-IIM is equivalent to w-IRM.

Lemma 17. $f : T \to \Delta(A)$ satisfies w-IIM on $\mathcal{T}$ if and only if f satisfies w-IRM on $\mathcal{T}$.

Proof. Assume that $f : T \to \Delta(A)$ satisfies w-IIM on $\mathcal{T}$. Take any unacceptable deception profile $\beta \in \mathcal{B}$ on $\mathcal{T}$ for f. Assume, to the contrary, that for all $(i, t_i, t_i') \in \mathcal{I} \times T_i \times \beta_i(t_i)$, there exists $\psi_i(t_i) \in \Delta_{t_i}^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}\right)$ such that for all $y_i^* \in \bar{Y}_i^f$, (41) is violated.[27] To derive a contradiction, let us first show that $\beta \subseteq \beta^{\alpha}$ for all $\alpha \in \Omega$. Let us proceed by transfinite induction.

By definition, $\beta \subseteq \bar{\beta} = \beta^0$. Fix an arbitrary $\alpha \in \Omega$ and suppose that for all $\gamma < \alpha$, it holds that $\beta \subseteq \beta^{\gamma}$. To complete the proof we need to show that $\beta \subseteq \beta^{\alpha}$. We proceed according to whether α is a limit ordinal or a successor ordinal. When α is a limit ordinal, the induction hypothesis and the definition of β^{α} implies that $\beta \subseteq \bigcap_{\gamma<\alpha} \beta^{\gamma} = \beta^{\alpha}$.

Suppose that α is a successor ordinal. Let us show that $\beta \subseteq \beta^{\alpha}$. By the inductive hypothesis, it holds that $\psi_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^{\alpha-1}}\left(\Theta \times T_{-i} \times \hat{T}\right)$ for all $i \in \mathcal{I}$ and all $t_i \in T_i$. Fix any $i \in \mathcal{I}$ and any $t_i \in T_i$. Take any $\hat{t}_i \in \beta_i(t_i)$. It follows from the inductive hypothesis that $\hat{t}_i \in \beta_i^{\alpha-1}(t_i)$. Since (41) is violated for $y_i^* \in \bar{Y}_i^f$, (42) implies that $\hat{t}_i \in \beta_i^{\alpha}(t_i)$. Since the triplet $\left(i, t_i, \hat{t}_i\right) \in \mathcal{I} \times T_i \times \beta_i(t_i)$ was chosen arbitrarily, we conclude that $\beta \subseteq \beta^{\alpha}$. By the principle of transfinite induction, it holds that $\beta \subseteq \beta^{\alpha}$ for all $\alpha \in \Omega$. Since Lemma 16 implies that the $(\beta^{\alpha})_{\alpha\in\Omega}$ is a monotonically decreasing net which converges to $\beta^* \in \mathcal{B}$, we have that $\beta \subseteq \beta^*$. Since f satisfies w-IIM on $\mathcal{T}$,

[27] Recall that $\bar{Y}^f$ is a nonempty metrizable subspace.

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it follows that β^* is an acceptable deception profile on $\mathcal{T}$ for f, and so β is also an acceptable deception profile on T for f, which is a contradiction.

Assume f satisfies w-IRM on $\mathcal{T}$. Assume, to the contrary, that $\beta^* \in \mathcal{B}$ is not acceptable on $\mathcal{T}$ for f. Since f satisfies w-IRM, it follows that there exists $(i, t_i, t_i') \in \mathcal{I} \times T_i \times \beta_i^*(t_i)$ such that for all $\psi_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^*}\left(\Theta \times T_{-i} \times \hat{T}\right)$, there exists $y_i^* \in \bar{Y}_i^f$ such that (41) is satisfied. Lemma 16 implies that there exists an $\alpha \in \Omega$ such that $\beta^\alpha = \beta^{\alpha+1} = \beta^*$. Since $t_i' \in \beta_i^*(t_i)$, (42) implies that there exists $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^*}\left(\Theta \times T_{-i} \times \hat{T}\right)$ such that

$$\begin{array}{ll}\sum_{(\theta,\hat{t})\in\Theta\times\hat{T}}\left(\text{marg}_{\Theta\times\hat{T}}\nu_i(t_i)\left[\theta,\hat{t}\right]\right)u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) & \geq \\ \sum_{(\theta,\hat{t})\in\Theta\times\hat{T}_{-i}}\left(\text{marg}_{\Theta\times\hat{T}}\nu_i(t_i)\left[\theta,\hat{t}\right]\right)u_i\left(y_i^*\left(\hat{t}_i,\hat{t}\right),\theta\right) & \end{array}$$

for all $y_i^* \in \bar{Y}_i^f$, yielding a contradiction. □

Any SCF satisfying our condition on $\mathcal{T}$ is *incentive compatible* on $\mathcal{T}$. The condition can be stated as follows.

Definition 14. $f : T \to \Delta(A)$ *incentive compatible* on $\mathcal{T}$ if for all $i \in \mathcal{I}$ and all $t_i \in T_i$,

$$\sum_{(\theta,t_{-i})\in\Theta\times T_{-i}}\kappa(t_i)[\theta,t_{-i}]\,u_i\left(f(t_i,t_{-i}),\theta\right) \geq \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}}\kappa(t_i)[\theta,t_{-i}]\,u_i\left(f(t_i',t_{-i}),\theta\right)$$

for all $t_i \in T_i$.

Lemma 18. $f : T \to \Delta(A)$ satisfies w-IIM on $\mathcal{T}$ implies that f is incentive compatible on $\mathcal{T}$.

Proof. It follows from Lemma 17 above and Lemma 5.2 of Kunimoto et al. (2020). □

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Necessity of w-IIM

Sufficiency of w-IIM

Assume that $f : T \to \Delta(A)$ satisfies w-IIM on $\mathcal{T}$. We show that $f : T \to \Delta(A)$ is ICR-implementable on $\mathcal{T}$. Before proving this result, we need additional notation. Fix any $\beta \in \mathcal{B}$, any $i \in \mathcal{I}$, and any $t_i \in T_i$. Let $\Delta_{t_i}^{\beta_{-i}}\left(\Theta \times \hat{T}\right)$ be defined by

$$\Delta_{t_i}^{\beta_{-i}}\left(\Theta \times \hat{T}\right) = \left\{ \psi_i \;\middle|\; \begin{array}{l} \text{There exists } \nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}\right) \\ \text{such that } \text{marg}_{\Theta \times \hat{T}} \nu_i(t_i) = \psi_i. \end{array} \right\} \tag{44}$$

The following definitions are critical in the construction of our implementing mechanism.

Definition 15. Let $\mathcal{T}$ be any model. For all $\beta \in \mathcal{B}$ and all $i \in \mathcal{I}$, $t_i \in T_i(\beta)$ if and only if for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}}\left(\Theta \times \hat{T}\right)$, there exist $y_i, \bar{y}_i \in \bar{Y}_i^f$ such that

$$\sum_{(\theta,\hat{t}) \in \Theta \times \hat{T}} \psi_i\left[\theta, \hat{t}\right] u_i\left(y_i\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right) > \sum_{(\theta,\hat{t}) \in \Theta \times \hat{T}} \psi_i\left[\theta, \hat{t}\right] u_i\left(\bar{y}_i\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right). \tag{45}$$

Definition 16. Let $\mathcal{T}$ be any model. For all $\beta \in \mathcal{B}$ and all $i \in \mathcal{I}$, $t_i \in T_i^*(\beta)$ if and only if there exist $\bar{y}_i \in \bar{Y}_i^f$ such that for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}}\left(\Theta \times \hat{T}\right)$, there exist $y_i \in \bar{Y}_i^f$ such that

$$\sum_{(\theta,\hat{t}) \in \Theta \times \hat{T}} \psi_i\left[\theta, \hat{t}\right] u_i\left(y_i\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right) > \sum_{(\theta,\hat{t}) \in \Theta \times \hat{T}} \psi_i\left[\theta, \hat{t}\right] u_i\left(\bar{y}_i\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right). \tag{46}$$

By adapting the arguments of the proof of Lemma 19, we show below that Definition 16 and Definition 15 are equivalent.

Lemma 19. Let $\mathcal{T}$ be any model. For all $\beta \in \mathcal{B}$, $T^*(\beta) = T(\beta)$.

Proof. Let $\mathcal{T}$ be any model. Fix any $\beta \in \mathcal{B}$ and $i \in \mathcal{I}$. Since it is clear that $T_i^*(\beta) \subseteq T_i(\beta)$, let us show that $T_i(\beta) \subseteq T_i^*(\beta)$. Assume that $t_i \in T_i(\beta)$. Definition 15 implies that for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}}\left(\Theta \times \hat{T}\right)$, there exist $y_i^{\psi_i}, \bar{y}_i^{\psi_i} \in \bar{Y}_i^f$ such that (45) is satisfied.

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Since $\Delta_{t_i}^{\beta_{-i}}\left(\Theta \times \hat{T}\right)$ is a separable metric space, let $\hat{\Delta}\left(\Theta \times \hat{T}\right) = \cup_{k\in\mathbb{N}} \{\psi_{i,k}\}$ be a countable, dense subset of $\Delta_{t_i}^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$. Let $\tilde{y}_i \in \bar{Y}_i^f$ be a mapping defined by

$$\tilde{y}_i = \sum_{k=1}^{\infty} \frac{1}{2^k} \bar{y}_i^{\psi_{i,k}}.$$

For all $\bar{k} \in \mathbb{N}$, let $y_i^{\psi_{i,\bar{k}}} \in \bar{Y}_i^f$ be a mapping defined by

$$y_i^{\bar{k}} = \sum_{k\neq\bar{k}} \frac{1}{2^k} \bar{y}_i^{\psi_{i,k}} + \frac{1}{2^{\bar{k}}} y_i^{\psi_{i,\bar{k}}}.$$

Thus, for all $k \in \mathbb{N}$, we have that

$$\sum_{\left(\theta,\hat{t}\right)\in\Theta\times\hat{T}} \psi_{i,k}\left[\theta, \hat{t}_{-i}\right] u_i\left(y_i^k\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right) > \sum_{\left(\theta,\hat{t}\right)\in\Theta\times\hat{T}} \psi_i\left[\theta, \hat{t}\right] u_i\left(\tilde{y}_i\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right),$$

where the strict inequality is guaranteed by (45). Since player i's preference over lotteries are continuous and since, moreover, $\hat{\Delta}\left(\Theta \times \hat{T}_{-i}\right)$ is a countable, dense subset of $\Delta_{t_i}^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, it follows that $t_i \in T^*(\beta)$. Since the choice of $i \in \mathcal{I}(\beta)$ was arbitrary, it follows that $T_i(\beta) \subseteq T_i^*(\beta)$. □

Lemma 20. Assume that $f : T \to \Delta(A)$ satisfies w-IIM on $\mathcal{T}$. For all $\alpha \in \Omega$, all $i \in \mathcal{I}$ and all $t_i \in T_i$, $t_i \in T_i^c(\beta^\alpha) \implies \beta_i^\alpha(t_i) = \beta_i^{\alpha+1}(t_i) = \bar{\beta}_i(t_i)$.[28]

Proof. Assume that $f : T \to \Delta(A)$ satisfies w-IIM on $\mathcal{T}$. Fix any $\alpha \in \Omega$. Assume that $t_i \in T_i^c(\beta^\alpha)$. Assume, to the contrary, that $\beta_i^{\alpha+1}(t_i) \neq \beta_i^\alpha(t_i)$. Since Lemma 16 implies that $(\beta^\alpha)_{\alpha\in\Omega}$ is a monotonic decreasing net, it follows that there exists $\left(t_i, \hat{t}_i\right)$ such that $\hat{t}_i \in \beta_i^\alpha(t_i)$ and $\hat{t}_i \notin \beta_i^{\alpha+1}(t_i)$. It follows from (42) that for all $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^\alpha}\left(\Theta \times T_{-i} \times \hat{T}\right)$,

$$\begin{array}{l} \sum_{\left(\theta,\hat{t}\right)\in\Theta\times\hat{T}_{-i}} \left(\text{marg}_{\Theta\times\hat{T}}\nu_i(t_i)\left[\theta, \hat{t}\right]\right) u_i\left(f\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right) \quad < \\ \sum_{\left(\theta,\hat{t}\right)\in\Theta\times\hat{T}} \left(\text{marg}_{\Theta\times\hat{T}}\nu_i(t_i)\left[\theta, \hat{t}\right]\right) u_i\left(\bar{y}_i\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right) \end{array}$$

[28] $T_i^c(\beta^\alpha) = \{t_i \in T_i | t_i \notin T_i(\beta^\alpha)\}$.

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for some $\bar{y}_i \in \bar{Y}_i^f$. Therefore, for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}^{\alpha}}\left(\Theta \times \hat{T}\right)$,

$$\sum_{(\theta,\hat{t})\in\Theta\times\hat{T}} \psi_i\left[\theta,\hat{t}\right] u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) < \sum_{(\theta,\hat{t})\in\Theta\times\hat{T}} \psi_i\left[\theta,\hat{t}\right] u_i\left(\bar{y}_i\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) \tag{47}$$

for some $\bar{y}_i \in \bar{Y}_i^f$. Let $y_i\left(\hat{t}_i,\hat{t}_{-i}\right) = f\left(\hat{t}_i,\hat{t}_{-i}\right)$ for every $\left(\hat{t}_i,\hat{t}_{-i}\right) \in \hat{T}$. Since f satisfies w-IIM on $\mathcal{T}$, Lemma 4 and Lemma 18 imply that f is incentive compatible on $\mathcal{T}$. This implies that $y_i(\cdot) \in \bar{Y}_i^f$. We have that the inequality in (45) holds for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}^{\alpha}}\left(\Theta \times \hat{T}\right)$. Definition 15 implies that $t_i \in T_i\left(\beta^{\alpha}\right)$, which is a contradiction.

Finally, let us show that $\beta_i^{\alpha+1}(t_i) = \beta_i^{\alpha}(t_i) = \bar{\beta}_i(t_i)$. Assume, to the contrary, that $\beta_i^{\alpha}(t_i) \neq \bar{\beta}_i(t_i)$. Since Lemma 16 implies that $\left(\beta_i^{\alpha}\right)_{\alpha\in\Omega}$ is a decreasing monotonic net, it follows that there exists a successor ordinal $\hat{\alpha}$ such that $0 < \hat{\alpha} \leq \alpha$ and that $\beta_i^{\hat{\alpha}}(t_i) \subseteq \beta_i^{\hat{\alpha}-1}(t_i)$ and $\beta_i^{\hat{\alpha}}(t_i) \neq \beta_i^{\hat{\alpha}-1}(t_i)$.[29] Thus, $\hat{t}_i \in \beta_i^{\hat{\alpha}-1}(t_i)$ and $\hat{t}_i \notin \beta_i^{\hat{\alpha}}(t_i)$ for some $\hat{t}_i \in T_i$. (42) implies that there exists $\bar{y}_i \in \bar{Y}_i^f$ such that

$$\begin{array}{l}\sum_{(\theta,\hat{t})\in\Theta\times\hat{T}}\left(\mathrm{marg}_{\Theta\times\hat{T}}\nu_i(t_i)\left[\theta,\hat{t}\right]\right) u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) \quad < \\ \sum_{(\theta,\hat{t})\in\Theta\times\hat{T}}\left(\mathrm{marg}_{\Theta\times\hat{T}}\nu_i(t_i)\left[\theta,\hat{t}\right]\right) u_i\left(\bar{y}_i\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right)\end{array}$$

for all $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^{\hat{\alpha}-1}}\left(\Theta \times T_{-i} \times \hat{T}\right)$. By definition of $\Delta_{t_i}^{\beta_{-i}^{\hat{\alpha}-1}}\left(\Theta \times \hat{T}\right)$ in (44), it follows that there exists $\bar{y}_i \in \bar{Y}_i^f$ such that

$$\sum_{(\theta,\hat{t})\in\Theta\times\hat{T}} \psi_i\left[\theta,\hat{t}\right] u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) < \sum_{(\theta,\hat{t})\in\Theta\times\hat{T}} \psi_i\left[\theta,\hat{t}\right] u_i\left(\bar{y}_i\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right)$$

for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}^{\hat{\alpha}-1}}\left(\Theta \times \hat{T}\right)$. Let $y_i\left(\hat{t}_i,\hat{t}_{-i}\right) = f\left(\hat{t}_i,\hat{t}_{-i}\right)$ for every $\left(\hat{t}_i,\hat{t}_{-i}\right) \in \hat{T}$. Since

[29] Suppose not. Then, for all successor ordinals $\hat{\alpha}$ such that $\hat{\alpha} \leq \alpha$, it holds that $\beta_i^{\hat{\alpha}}(t_i) = \beta_i^{\hat{\alpha}-1}(t_i)$. Suppose that $\beta_i^{\hat{\alpha}}(t_i) = \bar{\beta}_i(t_i)$ for all successor ordinals $\hat{\alpha}$ such that $\hat{\alpha} \leq \alpha$. It follows that for every limit ordinal $\delta \leq \alpha$, it holds that $\beta_i^{\delta}(t_i) = \bigcap_{\gamma<\delta}\beta_i^{\gamma}(t_i) = \bar{\beta}_i(t_i)$. An immediate contradiction is obtain if α is a limit ordinal. Thus, let α be a successor ordinal, and so $\beta_i^{\alpha}(t_i) = \bar{\beta}_i(t_i)$, which is a contradiction. Thus, there exists a successor ordinal α', with $\alpha' \leq \alpha$, such that $\beta_i^{\alpha'}(t_i) \neq \bar{\beta}_i(t_i)$. Since for all successor ordinals $\hat{\alpha}$ such that $\hat{\alpha} \leq \alpha$, it holds that $\beta_i^{\hat{\alpha}}(t_i) = \beta_i^{\hat{\alpha}-1}(t_i)$, it follows that for all successor ordinals $\hat{\alpha}$ such that $\hat{\alpha} \leq \alpha$, it holds that $\beta_i^{\hat{\alpha}}(t_i) \neq \bar{\beta}_i(t_i)$. Since $1 \in \Omega$ is a successor ordinal, it follows that there exists a successor ordinal such that $\beta_i^{1}(t_i) \subseteq \beta_i^{0}(t_i) = \bar{\beta}_i(t_i)$, yielding a contradiction.

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f satisfies w-IIM on $\mathcal{T}$, Lemma 4 and Lemma 18 imply that f is incentive compatible on $\mathcal{T}$. This implies that $y_i(\cdot) \in \bar{Y}_i^f$. Thus, Definition 15 implies that $t_i \in T_i\left(\beta^{\hat{\alpha}-1}\right)$. Since Lemma 16 implies that $(\beta_i^\alpha)_{\alpha\in\Omega}$ is a decreasing monotonic sequence and since, moreover, $\hat{\alpha}$ is such that $0 \neq \hat{\alpha} \leq \alpha$, it follows that

$$\sum_{(\theta,\hat{t})\in\Theta\times\hat{T}} \psi_i\left[\theta,\hat{t}\right] u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) < \sum_{(\theta,\hat{t})\in\Theta\times\hat{T}} \psi_i\left[\theta,\hat{t}\right] u_i\left(\bar{y}_i\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right)$$

for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}^{\alpha}}\left(\Theta\times\hat{T}\right) \subseteq \Delta_{t_i}^{\beta_{-i}^{\hat{\alpha}-1}}\left(\Theta\times\hat{T}\right)$. Definition 15 implies that $t_i \in T_i\left(\beta^\alpha\right)$, which is a contradiction. Thus, $\beta_i^{\alpha+1}(t_i) = \beta_i^\alpha(t_i) = \bar{\beta}_i(t_i)$. □

Lemma 21. Assume that $f : T \to \Delta(A)$ satisfies w-IIM on $\mathcal{T}$.

(i) If $T\left(\bar{\beta}\right) = \emptyset$, then f is constant.[30]

(ii) For all $i \in \mathcal{I}$, If $T_i(\beta^*) \neq T_i$, then for all $t_{-i} \in T_{-i}$ and all $t_i, t_i' \in T_i$, $f(t_i, t_{-i}) = f(t_i', t_{-i})$.

Proof. Assume that $f : T \to \Delta(A)$ satisfies w-IIM on $\mathcal{T}$. To show part (i), assume that $T\left(\bar{\beta}\right) = \emptyset$. Let us proceed by transfinite induction. We show that for all $i \in \mathcal{I}$, $\beta_i^\alpha(t_i) = \bar{\beta}_i(t_i)$ for all $t_i \in T_i$. The statement is clearly true for all $i \in \mathcal{I}$ if $\alpha = 0$. Thus, let $\alpha \neq 0$.

Suppose that α is a successor ordinal. Suppose that the statement is true for $\alpha - 1$ for all $i \in \mathcal{I}$ and all $t_i \in T_i$. Fix any $i \in \mathcal{I}$ and any $t_i \in T_i$. Assume, to the contrary, that $\beta_i^\alpha(t_i) \neq \bar{\beta}_i(t_i) = T_i$. Then, there exists $\hat{t}_i \in T_i$ such that $\hat{t}_i \notin \beta_i^\alpha(t_i)$ and $\hat{t}_i \in \beta_i^{\alpha-1}(t_i) = \bar{\beta}_i(t_i)$. It follows from (42) that for all $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^{\alpha-1}}\left(\Theta\times T_{-i}\times\hat{T}\right)$,

$$\begin{array}{l}\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}}\left(\text{marg}_{\Theta\times\hat{T}}\nu_i(t_i)\left[\theta,\hat{t}\right]\right) u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) \quad < \\ \sum_{(\theta,\hat{t})\in\Theta\times\hat{T}}\left(\text{marg}_{\Theta\times\hat{T}}\nu_i(t_i)\left[\theta,\hat{t}\right]\right) u_i\left(\bar{y}_i\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right)\end{array}$$

for some $\bar{y}_i \in \bar{Y}_i^f$. By definition of $\Delta_{t_i}^{\beta_{-i}^{\alpha-1}}\left(\Theta\times\hat{T}\right)$ in (44), it follows that there exists $\bar{y}_i \in \bar{Y}_i^f$ such that

[30] f is constant if for all $t, t' \in T$, $f(t) = f(t')$.

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$$\sum_{(\theta,\hat{t})\in\Theta\times\hat{T}} \psi_i\left[\theta,\hat{t}\right] u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) < \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}} \psi_i\left[\theta,\hat{t}\right] u_i\left(\bar{y}_i\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right)$$

for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}^{\alpha-1}}\left(\Theta\times\hat{T}\right)$. Let $y_i\left(\hat{t}_i,\hat{t}_{-i}\right) = f\left(\hat{t}_i,\hat{t}_{-i}\right)$ for all $\left(\hat{t}_i,\hat{t}_{-i}\right)\in\hat{T}$. Since f satisfies w-IIM on $\mathcal{T}$, Lemma 4 and Lemma 18 imply that f is incentive compatible on $\mathcal{T}$. This implies that $y_i(\cdot)\in\bar{Y}_i^f$. Since the choice of $t_i\in T_i$ was arbitrary and since $\beta_{-i}^{\alpha-1}(t_{-i}) = \bar{\beta}_{-i}(t_{-i})$ for all $t_{-i}\in T_{-i}$, we have that $t_i\in T_i\left(\bar{\beta}\right)$, which is a contradiction. Thus, we conclude that $\beta_i^{\alpha}(t_i) = \bar{\beta}_i(t_i)$ for all $i\in\mathcal{I}$ and all $t_i\in T_i$.

Suppose that $\alpha\neq 0$ is a limit ordinal. Suppose that for all $i\in\mathcal{I}$ and all $t_i\in T_i$, it holds that $\beta_i^{\gamma}(t_i) = \bar{\beta}_i(t_i)$. Since, by definition, $\beta_i^{\alpha}(t_i) = \bigcap_{\gamma<\alpha}\beta_i^{\gamma}(t_i)$ for all $i\in\mathcal{I}$ and all $t_i\in T_i$, it follows that $\beta_i^{\alpha}(t_i) = \bar{\beta}_i(t_i)$ for all $i\in\mathcal{I}$ and all $t_i\in T_i$.

Since β^* is the limit point of $(\beta^{\alpha})_{\alpha\in\Omega}$, it follows that $\beta_i^*(t_i) = \bar{\beta}_i(t_i)$ for all $i\in\mathcal{I}$ and all $t_i\in T_i$. Fix any $t^*\in T$. Since f satisfies w-IIM on $\mathcal{T}$, it follows that for all $t\in\beta^*(t^*) = T$, it holds that $f(t) = f(t^*)$. Thus, f is constant. This completes the proof of part (i).

Let us show part (*ii*). Fix any $i\in\mathcal{I}$ such that $T_i(\beta^*)\neq T_i$. Let us show that $f(t_i,t_{-i}) = f(t_i',t_{-i})$ for all $t_i,t_i'\in T_i$ and $t_{-i}\in T_{-i}$. Since f satisfies w-IIM on $\mathcal{T}$, it is enough to show that $\beta_i^*(t_i) = T_i$ for some $t_i\in T_i$.[31] Assume, to the contrary, that $\beta_i^*(t_i)\neq T_i$ for all $t_i\in T_i$. Fix any successor ordinal α such that $\beta_i^*(t_i) = \beta_i^{\alpha}(t_i) = \beta_i^{\alpha-1}(t_i)$ for all $t_i\in T_i$. It follows from (42) that for all $t_i\in T_i$ and all $\nu_i(t_i)\in\Delta_{t_i}^{\beta_{-i}^{\alpha-1}}\left(\Theta\times T_{-i}\times\hat{T}\right)$,

$$\begin{aligned}&\textstyle\sum_{(\theta,\hat{t})\in\Theta\times\hat{T}}\left(\mathrm{marg}_{\Theta\times\hat{T}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right)u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) \qquad <\\ &\textstyle\sum_{(\theta,\hat{t})\in\Theta\times\hat{T}}\left(\mathrm{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right)u_i\left(\bar{y}_i\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right)\end{aligned}$$

for some $\bar{y}_i\in\bar{Y}_i^f$. By definition of $\Delta_{t_i}^{\beta_{-i}^{\alpha-1}}\left(\Theta\times\hat{T}\right)$ in (44), it follows that there exists

[31] To see it, suppose that $\beta^*(t_i) = T_i$ for some $t_i\in T_i$. Fix any $t_{-i}\in T_i$. Since β^* is an acceptable deception, it follows that $f(t_i',t_{-i}) = f(t_i'',t_{-i})$ for all $t_i',t_i''\in\beta_i^*(t_i)$.

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$\bar{y}_i \in \bar{Y}_i^f$ such that

$$\sum_{(\theta,\hat{t})\in\Theta\times\hat{T}} \psi_i\left[\theta,\hat{t}\right] u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) < \sum_{(\theta,\hat{t})\in\Theta\times\hat{T}} \psi_i\left[\theta,\hat{t}\right] u_i\left(\bar{y}_i\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right)$$

for all $\psi_i \in \Delta_{t_i}^{\beta_{-i}^{\alpha-1}}\left(\Theta\times\hat{T}\right)$ and all $t_i \in T_i$. Let $y_i\left(\hat{t}_i,\hat{t}_{-i}\cdot\right) = f\left(\hat{t}_i,\hat{t}_{-i}\right)$ for all $\hat{t}_i,\hat{t}_{-i} \in \hat{T}$. Since f satisfies w-IIM on $\mathcal{T}$, Lemma 4 and Lemma 18 imply that f is incentive compatible on $\mathcal{T}$. This implies that $y_i(\cdot) \in \bar{Y}_i^f$. Definition 15 implies that $T_i(\beta^*) = T_i$, which is a contradiction. This completes the proof of part (ii). □

In what follows, to avoid trivialities, we assume that $T_i\left(\bar{\beta}\right) \neq \emptyset$ for all $i \in \mathcal{I}$. Moreover, we assume that $T_i(\beta^*) = T_i$ for all $i \in \mathcal{I}$.

Lemma 22. For all $(\alpha,i,t_i) \in \Omega\times\mathcal{I}\times T_i$, if $t_i \in T_i(\beta^\alpha)\backslash T_i(\beta^0)$, then there exists $\hat{\alpha}(t_i) \leq \alpha$ such that $t_i \in T_i\left(\beta^{\hat{\alpha}(t_i)}\right)$ and $t_i \in T_i^c(\beta^\gamma)$ for all $\gamma < \hat{\alpha}(t_i)$.

Proof. Fix any $(\alpha,i,t_i) \in \Omega\times\mathcal{I}\times T_i$ such that $t_i \in T_i(\beta^\alpha)\backslash T_i(\beta^0)$. Assume, to the contrary, that there does not exist any $\hat{\alpha} \in \Omega$, with $\hat{\alpha} \leq \alpha$, such that $t_i \in T_i\left(\beta^{\hat{\alpha}}\right)$ and $t_i \in T_i^c(\beta^\gamma)$ for all $\gamma < \hat{\alpha}$. Thus, for all $\hat{\alpha} \in \Omega$, with $\hat{\alpha} \leq \alpha$, it holds that $t_i \in T_i^c\left(\beta^{\hat{\alpha}}\right)$ or $t_i \in T_i(\beta^\gamma)$ for some $\gamma < \hat{\alpha}$.

Suppose that there exists $\hat{\alpha} \in \Omega$, with $\hat{\alpha} \leq \alpha$, such that $t_i \in T_i(\beta^\gamma)$ for some $\gamma < \hat{\alpha}$. Let us consider the set $\bar{\Omega} = \left\{\delta \in \Omega\backslash\{0\} \,|\delta \leq \gamma < \hat{\alpha} \text{ and } t_i \in T_i\left(\beta^\delta\right)\right\}$. Let $\gamma^* \in \bar{\Omega}$ be such that $\gamma^* \leq \delta$ for all $\delta \in \bar{\Omega}$. We have that $t_i \in T_i\left(\beta^{\gamma^*}\right)$ and $t_i \in T_i^c(\beta^\gamma)$ for all $\gamma < \gamma^*$, which is a contradiction. Otherwise, suppose that for all $\hat{\alpha} \in \Omega$, with $\hat{\alpha} \leq \alpha$, $t_i \in T_i^c(\beta^\gamma)$ for all $\gamma < \hat{\alpha}$. Since $t_i \in T_i(\beta^\alpha)$, it follows that $t_i \in T_i(\beta^\alpha)$ and $t_i \in T_i^c(\beta^\gamma)$ for all $\gamma < \alpha$, which is a contradiction. □

The following result is useful in defining *Rule 3* of the mechanism.

Lemma 23. Let $\mathcal{T}$ be any model. For all $i \in \mathcal{I}$ such that $T_i(\beta^*) \neq \emptyset$, there exists $\hat{y}_i \in \Delta(A)$ such that for every $\phi_i \in \Delta(\Theta)$, there exists $y_i \in \Delta(A)$ such that

$$\sum_{\theta\in\Theta} \phi_i(\theta)\, u_i(y_i,\theta) > \sum_{\theta\in\Theta} \phi_i(\theta)\, u_i(\hat{y}_i,\theta). \tag{48}$$

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Proof. Fix any $i \in \mathcal{I}$ and any $t_i \in T_i(\beta^*)$. Lemma 19 implies that $t_i \in T_i^*(\beta^*)$. Definition 16 implies that there exists $\bar{y}_i \in \bar{Y}_i^f$ such that for all $\psi_i \in \Delta_{t_i}^{\beta^*_{-i}}\left(\Theta \times \hat{T}\right)$, there exists $y_i \in \bar{Y}_i^f$ such that (46) holds. Since $\beta^t \subseteq \beta^*$, it follows that there exists $\bar{y}_i \in \bar{Y}_i^f$ such that for all $\psi_i \in \Delta_{t_i}^{\beta^t_{-i}}\left(\Theta \times \hat{T}\right)$, there exists $y_i \in \bar{Y}_i^f$ such that (46) holds. Fix any $t_i \in T_i$. Observe that $\phi_i \circ \left(\text{marg}_{T_{-i}}\kappa(t_i)\right) \in \Delta_{t_i}^{\beta^t_{-i}}\left(\Theta \times \hat{T}\right)$ for all $\phi_i \in \Delta(\Theta)$. Therefore, it holds that

$$\sum_{\left(\theta,\hat{t}\right)\in\Theta\times\hat{T}} \left(\phi_i[\theta]\left(\text{marg}_{T_{-i}}\kappa(t_i)\left[\hat{t}_{-i}\right]\right)\right)\left[u_i\left(y_i\left(\hat{t}_{-i}\right),\theta\right) - u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)\right] > 0.$$

By setting

$$y_i = \sum_{\hat{t}_{-i}\in\hat{T}_{-i}} \left(\text{marg}_{T_{-i}}\kappa(t_i)\left[\hat{t}_{-i}\right]\right) y_i\left(\hat{t}_{-i}\right)$$

and

$$\hat{y}_i = \sum_{\hat{t}_{-i}\in\hat{T}_{-i}} \left(\text{marg}_{T_{-i}}\kappa(t_i)\left[\hat{t}_{-i}\right]\right) \bar{y}_i\left(\hat{t}_{-i}\right),$$

and by noting that $y_i, \hat{y}_i \in \Delta(A)$, the inequality in (48) follows for i. Since the choice of $i \in \mathcal{I}$ such that $T_i(\beta^*) \neq \emptyset$ was arbitrary, the statement follows. □

Let $\mathcal{T}$ be any model. Since $T_i(\beta^*) = T_i$ for all $i \in \mathcal{I}$ and since Lemma 48 guarantees the existence of the lottery $\hat{y}_i \in \Delta(A)$ for all $i \in \mathcal{I}$, let us define the lottery $\hat{y}$ by

$$\hat{y} = \frac{1}{I}\sum_{i\in\mathcal{I}} \hat{y}_i.$$

Given the net $(\beta^\alpha)_{\alpha\in\Omega}$ and our assumption that $T(\beta^*) = T$, Lemma 16 implies that for some $\alpha \in \Omega$, it holds that $T(\beta^\alpha) = T$. Fix any $i \in \mathcal{I}$ and any $t_i \in T_i$. Then, if $t_i \in T_i \setminus T_i(\beta^0)$, Lemma 22 implies that there exists a least ordinal $\alpha(t_i)$ such that $t_i \in T_i\left(\beta^{\alpha(t_i)}\right)\setminus T_i(\beta^\gamma)$ for every $\gamma < \alpha(t_i)$. Otherwise, $\alpha(t_i) = 0$. Therefore, for all $t_i \in T_i$, there exists a least ordinal $\alpha(t_i)$ such that $t_i \in T_i\left(\beta^{\alpha(t_i)}\right)\setminus T_i(\beta^\gamma)$ for every $\gamma < \alpha(t_i)$. Lemma 19 implies that for all $t_i \in T_i$, there exists a least ordinal $\alpha(t_i)$ such that $t_i \in T_i^*\left(\beta^{\alpha(t_i)}\right)\setminus T_i^*(\beta^\gamma)$ for every $\gamma < \alpha(t_i)$. Since $t_i \in T_i^*\left(\beta^{\alpha(t_i)}\right)$, Definition 16 implies that there exists $\bar{y}_i \in \bar{Y}_i^f$ satisfying (46). Let us denote $\bar{y}_i$ by $\bar{y}_i^{\beta^{\alpha(t_i)}}$. Define

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the allocation $\bar{y}_i^{\beta^{\bar{\alpha}(i)}}$ as follows:

$$\bar{y}_i^{\beta^{\bar{\alpha}(i)}} = \sum_{l \geq 0} \frac{1}{2^l} \bar{y}_i^{\beta^{\alpha(t_i^l)}} \tag{49}$$

where $l \geq 0$ is an enumeration of T_i. Since $\bar{y}_i^{\beta^{\bar{\alpha}(i)}} \in Y_{i,s}^f$, we can choose an $\varepsilon > 0$ sufficiently small such that the mapping $\mathfrak{y}_i^{\beta^{\bar{\alpha}(i)}} : T \to \Delta(A)$ defined by

$$\mathfrak{y}_i^{\beta^{\bar{\alpha}(i)}}(t) = (1-\varepsilon)\, \bar{y}_i^{\beta^{\bar{\alpha}(i)}}(t) + \varepsilon \hat{y} \tag{50}$$

is such that $\mathfrak{y}_i^{\beta^{\bar{\alpha}(i)}} \in Y_{i,s}^f$.

Before stating our mechanism, we need the following lemma.

Lemma 24. For all $i \in \mathcal{I}$ and all $t_i \in T_i$, if $t_i \in T_i^*\left(\beta^{\alpha(t_i)}\right) \backslash T_i^*(\beta^\gamma)$ for every $\gamma < \alpha(t_i)$, then for all $\psi_i \in \Delta_{t_i}^{\beta^\alpha}(\Theta \times \hat{T})$, with $\alpha \geq \alpha(t_i)$, there exists $y_i' \in Y_i^f$ such that

$$\sum_{(\theta,\hat{t}) \in \Theta \times \hat{T}} \psi_i(\theta,\hat{t}) u_i(y_i'(\hat{t}_i,\hat{t}_{-i}),\theta) > \sum_{(\theta,\hat{t}) \in \Theta \times \hat{T}} \psi_i(\theta,\hat{t}) u_i(\mathfrak{y}_i^{\beta^{\bar{\alpha}(i)}}(\hat{t}_i,\hat{t}_{-i}),\theta). \tag{51}$$

Proof. Fix any $i \in \mathcal{I}$ and any $t_i \in T_i$. Suppose that $t_i \in T_i^*\left(\beta^{\alpha(t_i)}\right) \backslash T_i^*(\beta^\gamma)$ for every $\gamma < \alpha(t_i)$. Definition 16 implies that for all $\psi_i \in \Delta_{t_i}^{\beta^{\alpha(t_i)}}(\Theta \times \hat{T}_{-i})$, there exists $y_i \in Y_i^f$ such that

$$\sum_{(\theta,\hat{t}_{-i}) \in \Theta \times \hat{T}_{-i}} \psi_i(\theta,\hat{t}_{-i}) u_i(y_i(\hat{t}_{-i}),\theta) > \sum_{(\theta,\hat{t}_{-i}) \in \Theta \times \hat{T}_{-i}} \psi_i(\theta,\hat{t}_{-i}) u_i(\bar{y}_i^{\beta^{\alpha(t_i)}}(\hat{t}_{-i}),\theta).$$

Since $\bar{y}_i^{\beta^{\alpha(t_i)}} \in$Supp$(\bar{y}_i^{\beta^{\bar{\alpha}(i)}})$, we can see that there exists $y_i' \in Y_i^f$ such that the inequality in the statement holds for all $\psi_i \in \Delta_{t_i}^{\beta^{\alpha(t_i)}}(\Theta \times \hat{T}_{-i})$. Since Lemma 16 implies that $\beta^\alpha \subseteq \beta^{\alpha(t_i)}$ for all $\alpha \in \Omega$ such that $\alpha(t_i) \geq \alpha$, we can see that the inequality (65) in the statement holds for all $\psi_i \in \Delta_{t_i}^{\beta^\alpha}(\Theta \times \hat{T}_{-i})$ with $\alpha(t_i) \geq \alpha$. □

Let us now define the mechanism $\mathcal{M}$. For the sake of brevity, we focus on the case

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that $I = 2$. [32] For all $i \in \mathcal{I}$, let

$$M_i = M_i^1 \times M_i^2 \times M_i^3 \times M_i^4,$$

where

$$M_i^1 = T_i \times T_{-i},\ M_i^2 = \mathbb{N},\ M_i^3 = Y_i^* \text{ and } M_i^4 = \Delta^*(A),$$

where $\mathbb{N}$ is the set of natural numbers, Y_i^* is a countable, dense subset of $\bar{Y}_i^f$, and $\Delta^*(A)$ is a countable, dense subset of $\Delta(A)$. Let $proj_{T_i} m_i^1 = m_i^{11}$. For all $m \in M$, let $g : M \to \Delta(A)$ be defined as follows.

Rule 1: If $m_i^2 = 1$ for all $i \in \mathcal{I}$, then $g(m) = f\left(m_i^{11}, m_{-i}^{11}\right)$.

Rule 2: For all $i \in \mathcal{I}$, if $m_j^2 = 1$ for all $j \in \mathcal{I} \setminus \{i\}$ and $m_i^2 > 1$, then

$$g(m) = m_i^3\left(m_{-i}^1\right)\left(1 - \frac{1}{1 + m_i^2}\right) \oplus \mathfrak{y}_i^{\beta^{\bar{\alpha}(i)}}\left(m_{-i}^1\right)\left(\frac{1}{1 + m_i^2}\right), \tag{52}$$

where $\mathfrak{y}_i^{\beta^{\bar{\alpha}(i)}} \in Y_{i,s}^f$ is defined in (50).

Rule 3: Otherwise, for each $i \in \mathcal{I}$, m_i^4 is picked with probability $\frac{1}{I}\left(1 - \frac{1}{1+m_i^2}\right)$ and $\hat{y}_i$ is picked with probability $\frac{1}{I}\left(\frac{1}{1+m_i^2}\right)$; that is,

$$g(m) = \frac{1}{I}\left[m_i^4\left(1 - \frac{1}{1 + m_i^2}\right) \oplus \hat{y}_i\left(\frac{1}{1 + m_i^2}\right)\right], \tag{53}$$

where $\hat{y}_i$ is specified by Lemma 23.

Suppose that f satisfies w-IIM on $\mathcal{T}$. In what follows, we prove that $\mathcal{M}$ ICR-implements f on $\mathcal{T}$. The following lemmata will help us to complete the proof.

Lemma 25. For all $i \in \mathcal{I}$ and all $t_i \in T_i$, $S_i^{\mathcal{M},\mathcal{T}}(t_i) \neq \emptyset$.

Proof. For every $i \in \mathcal{I}$ and every $t_i \in T_i$, let $\Sigma_i(t_i)$ be defined as

[32] The mechanism for the case $I \geq 2$ can be obtained by modifying Rule 2 below in a way that if the opponents of player i announce different type profiles, then we choose the profile selected by the winner of the modulo game.

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$$\Sigma_i(t_i) = \left\{ m_i \in M_i \middle| m_i^{11} = t_i, m_i^2 = 1 \right\} \tag{54}$$

We will show that Σ is a best reply set in $(U(\mathcal{M},\mathcal{T}))$. Towards this end, fix an agent i and t_i. Let $\sigma_j : T_j \to M_j$ be defined by $\sigma_j(t_j) = ((t_j, t_i), 1, \cdot, \cdot)$. For all $i \in \mathcal{I}$ and all $t_i \in T_i$, let $\pi_i(t_i) \in \Delta(\Theta \times T_{-i} \times M_{-i})$ be defined by

$$\pi_i(t_i)[\theta, t_{-i}, m_{-i}] = \kappa(t_i)[\theta, t_{-i}]\,\delta_{\sigma_{-i}(t_{-i})}[m_{-i}],$$

where $\delta_{\sigma_{-i}(t_{-i})}$ is the dirac measure on $\{\sigma_{-i}(t_{-i})\}$. By construction, for all $t_i \in T_i$ and all $(\theta, t_{-i}, m_{-i}) \in \Theta \times T_i \times M_{-i}$, $\pi_i(t_i)[\theta, t_i, m_{-i}] > 0 \implies m_{-i} = \sigma_{-i}(t_{-i})$.

To see this, by construction and *Rule 1*, for all $i \in \mathcal{I}$, $t_i \in T_i$, and $m_i \in \Sigma_i(t_i)$

$$\begin{aligned} & \sum_{(\theta, m_{-i}) \in \Theta \times M_{-i}} \operatorname{marg}_{\Theta \times M_{-i}} \pi_i(t_i)[\theta, m_{-i}]\, u_i(g(m_i, m_{-i}), \theta) \\ = & \sum_{(\theta, t_{-i}) \in \Theta \times T_{-i}} \kappa(t_i)[\theta, t_{-i}]\, u_i(f(t_i, t_{-i}), \theta). \end{aligned}$$

Finally, by definition of g and the fact that f is incentive compatible on $\mathcal{T}$ (Lemma 18), it follows that for all $i \in \mathcal{I}$ and all $t_i \in T_i$, $\Sigma_i(t_i) \subseteq BR_i\left(\operatorname{marg}_{\Theta \times M_{-i}} \pi_i(t_i) | \mathcal{M}\right)$. Since $\sigma_j(t_j) \in \Sigma_j(t_j)$ and i was chosen arbitrarily, we have shown that Σ is a best reply set in $(U(\mathcal{M},\mathcal{T}))$ and so $S_i^{\mathcal{M},\mathcal{T}}(t_i) \neq \emptyset$ for every $i \in \mathcal{I}$ and $t_i \in T_i$.

□

Before proving the following lemma, let us introduce the following definitions. For all $\beta \in \mathcal{B}$ and all $i \in \mathcal{I}$, define $\Sigma_i^{\beta_i} : T_i \to 2^{M_i} \setminus \{\emptyset\}$ by

$$\Sigma_i^{\beta_i}(t_i) = \left\{ m_i \in M_i | m_i^{11} \in \beta_i(t_i) \right\}, \tag{55}$$

and define $\tilde{\Sigma}_i^{\beta_i} : T_i \to 2^{M_i} \setminus \{\emptyset\}$ by

$$\tilde{\Sigma}_i^{\beta_i}(t_i) = \left\{ m_i \in \Sigma_i^{\beta_i}(t_i) | m_i^2 = 1 \right\}. \tag{56}$$

It can be checked that $\Sigma^{\beta}, \tilde{\Sigma}^{\beta} \in \mathfrak{S}^{\mathcal{M},\mathcal{T}}$.

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Lemma 26. For all $(\alpha, i, t_i) \in \Omega \times \mathcal{I} \times T_i(\beta^\alpha)$ and all $\pi_i(t_i) \in \Delta_{t_i}^{\Sigma_{-i}^{\beta_{-i}^{\alpha}}}(\Theta \times T_{-i} \times M_{-i})$, if

$$m_i \in BR_i\left(\text{marg}_{\Theta \times M_{-i}} \pi_i(t_i) | \mathcal{M}\right) \tag{57}$$

then $m_i^2 = 1$, $\pi_i(t_i) \in \Delta_{t_i}^{\tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}}(\Theta \times T_{-i} \times M_{-i})$ and $m_i^{11} \in \beta_i^{\alpha+1}(t_i)$.

Proof. Fix any $(\alpha, i, t_i) \in \Omega \times \mathcal{I} \times T_i(\beta^\alpha)$. Suppose that $\pi_i(t_i) \in \Delta_{t_i}^{\Sigma_{-i}^{\beta_{-i}^{\alpha}}}(\Theta \times T_{-i} \times M_{-i})$ and that $m_i \in BR_i\left(\text{marg}_{\Theta \times M_{-i}} \pi_i(t_i) | \mathcal{M}\right)$. Let us first show that $m_i^2 = 1$. Assume, to the contrary, that $m_i^2 > 1$. Let us proceed according to whether *Rule 2* applies or *Rule 3* applies. To this end, let us note that $\pi_i(t_i) \in \Delta_{t_i}^{\Sigma_{-i}^{\beta_{-i}^{\alpha}}}(\Theta \times T_{-i} \times M_{-i})$ can be decomposed as follows:

$$\underbrace{\sum_{\theta \in \Theta} \sum_{t_{-i} \in T_{-i}} \sum_{m_{-i} \in \tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}(t_{-i})} \pi_i(t_i)[\theta, t_{-i}, m_{-i}]}_{Prob[Rule2]} + \underbrace{\sum_{\theta \in \Theta} \sum_{t_{-i} \in T_{-i}} \sum_{m_{-i} \in M_{-i} \setminus \tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}(t_{-i})} \pi_i(t_i)[\theta, t_{-i}, m_{-i}]}_{Prob[Rule3]} = 1. \tag{58}$$

For all $i \in \mathcal{I}$ and all $t_i \in T_{-i}$, define $\nu_i(t_i) \in \Delta(\Theta \times T_{-i} \times M_{-i}^1)$ by

$$\nu_i(t_i)[\theta, t_{-i}, m_{-i}^1] = \frac{\sum\limits_{m_{-i} \in \tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}(t_{-i})[m_{-i}^1]} \pi_i(t_i)[\theta, t_{-i}, m_{-i}]}{Prob[Rule2]}. \tag{59}$$

Since $\pi_i(t_i) \in \Delta_{t_i}^{\Sigma_{-i}^{\beta_{-i}^{\alpha}}}(\Theta \times T_{-i} \times M_{-i})$, it follows that $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^{\alpha}}(\Theta \times T_{-i}^1 \times M_{-i}^1)$. Let $\psi_i = marg_{\Theta \times M_{-i}^1} \nu_i(t_i)$. Since $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^{\alpha}}(\Theta \times T_{-i}^1 \times M_{-i}^1)$, it holds that

$$\psi_i \in \Delta_{t_i}^{\beta_{-i}^{\alpha}}(\Theta \times M_{-i}^1). \tag{60}$$

Next, let $\phi_i(\theta) \in \Delta(\Theta)$ be defined by

$$\phi_i(\theta) = \frac{\sum\limits_{t_{-i} \in T_{-i}} \sum\limits_{m_{-i} \in M_{-i} \setminus \tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}(t_{-i})} \pi_i(t_i)[\theta, t_{-i}, m_{-i}]}{Prob[Rule3]}. \tag{61}$$

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The utility of m_i under the beliefs $marg_{\Theta\times M_{-i}}\pi_i(t_i)$, which is denoted by $U_i(m_i, marg_{\Theta\times M_{-i}}\pi_i(t_i))$, is given by

$$U_i(m_i, marg_{\Theta\times M_{-i}}\pi_i(t_i)) = \alpha \sum_{(\theta,t)\in\Theta\times\hat{T}} \psi_i(\theta,t)u_i\Big[\Big(1-\frac{1}{m_i^2+1}\Big)m_i^3(t) \oplus \frac{1}{m_i^2+1}\mathfrak{y}_i^{\beta^{\bar{\alpha}(i)}}(t)\Big),\theta\Big]$$
$$+ (1-\alpha)\sum_{\theta\in\Theta}\phi_i(\theta)u_i\Big[\Big(1-\frac{1}{m_i^2+1}\Big)m_i^4 \oplus \frac{1}{m_i^2+1}\hat{y}_i\Big),\theta\Big] \tag{62}$$

where $\alpha = Prob[Rule2]$.

Since $\psi_i \in \Delta_{t_i}^{\beta^{\alpha}_{-i}}(\Theta\times\hat{T})$ and $t_i \in T_i(\beta^{\alpha})$ and since $\alpha \geq \alpha(t_i)$, Lemma 24 implies that there exists $y_i' \in Y_i^f$ such that

$$\sum_{(\theta,\hat{t})\in\Theta\times\hat{T}} \psi_i(\theta,\hat{t})u_i(y_i'(\hat{t}_i,\hat{t}_{-i}),\theta) > \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}} \psi_i(\theta,\hat{t})u_i(\mathfrak{y}_i^{\beta^{\bar{\alpha}(i)}}(\hat{t}_i,\hat{t}_{-i}),\theta). \tag{63}$$

Furthermore, Lemma 23 implies that there exists $y_i \in \Delta(A)$ such that

$$\sum_{\theta\in\Theta}\phi_i(\theta)u_i(y_i,\theta) > \sum_{\theta\in\Theta}\phi_i(\theta)u_i(\hat{y}_i,\theta). \tag{64}$$

Since $m_i \in BR_i(marg_{\Theta\times M_{-i}}\pi_i(t_i)|\mathcal{M})$, it follows that

$$\sum_{(\theta,\hat{t})\in\Theta\times\hat{T}} \psi_i(\theta,\hat{t})u_i(m_i^3(\hat{t}),\theta) \geq \sum_{(\theta,\hat{t})\in\Theta\times\hat{T}} \psi_i(\theta,\hat{t})u_i(y_i'(\hat{t}),\theta) \tag{65}$$

and that

$$\sum_{\theta\in\Theta}\phi_i(\theta)u_i(m_i^4,\theta) \geq \sum_{\theta\in\Theta}\phi_i(\theta)u_i(y_i,\theta). \tag{66}$$

Inequalities in (63)-(66) imply that $U_i(m_i, marg_{\Theta\times M_{-i}}\pi_i(t_i))$ is strictly increasing in m_i^2, which is a contradiction. Thus, $m_i^2 = 1$.

Next, let us show that $\pi_i(t_i) \in \Delta_{t_i}^{\tilde{\Sigma}_{-i}^{\beta^{\alpha}_{-i}}}(\Theta\times T_{-i}\times M_{-i})$. Assume, to the contrary, that $\pi_i(t_i) \notin \Delta_{t_i}^{\tilde{\Sigma}_{-i}^{\beta^{\alpha}_{-i}}}(\Theta\times T_{-i}\times M_{-i})$. Then, since $m_i^2 = 1$, either *Rule 2* applies where

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$m_j^2 > 1$ for some $j \in \mathcal{I} \backslash \{i\}$ or *Rule 3* applies. In what follows, we focus only on the case that *Rule 2* applies.[33]

By the definition of g, for all $(\theta, m_{-i}) \in$Supp$(\underset{\Theta \times M_{-i}}{marg} \pi_i(t_i))$, it holds that

$$g(m_i, m_{-i}) = (1 - \frac{1}{m_j^2 + 1}) m_j^3(m_{-j}^1) + \frac{1}{m_j^2 + 1} \mathfrak{y}_j^{\beta^{\bar{\alpha}(j)}}(m_{-j}^1), \tag{67}$$

where, for $\varepsilon > 0$ sufficiently small,

$$\mathfrak{y}_j^{\beta^{\bar{\alpha}(j)}}(t) = (1 - \varepsilon) \bar{y}_j^{\beta^{\bar{\alpha}(j)}}(t) + \varepsilon \hat{y}. \tag{68}$$

To show that player i can gain by triggering *Rule 3*, we need to define a lottery $\hat{m}_i^4 \in \Delta^*(A) = M_i^4$ that can be used by player i. To this end, we first define the allocation h over M as follows: For all (m_i, m_{-i}) such that $(\theta, m_{-i}) \in$Supp$(\underset{\Theta \times M_{-i}}{marg} \pi_i(t_i))$,

$$h(m_i, m_{-i}) = (1 - \frac{1}{m_j^2 + 1}) m_j^3(m_{-j}^1) + \frac{1}{m_j^2 + 1} \tilde{\mathfrak{y}}_j^{\beta^{\bar{\alpha}(j)}}(m_{-j}^1) \tag{69}$$

where $\tilde{\mathfrak{y}}_j^{\beta^{\bar{\alpha}(j)}}(t_{-j})\, (1 - \varepsilon) \bar{y}_j^{\beta^{\bar{\alpha}(j)}}(t_{-j}) + \varepsilon [\sum_{j \neq i} \frac{1}{I} \hat{y}_j + \frac{1}{I} y_i]$ and y_i is such that (48) is satisfied. Finally, let us define $\hat{m}_i^4$ by

$$\hat{m}_i^4 = \sum marg_{\Theta \times M_{-i}} \pi_i(t_i)(\theta, m_{-i}) h(\cdot, m_{-i}). \tag{70}$$

Since player i's utility is strictly higher under $h(m_i, m_{-i})$ than under $g(m_i, m_{-i})$ for each $(\theta, m_{-i}) \in$Supp$(\underset{\Theta \times M_{-i}}{marg} \pi_i(t_i))$ and since, moreover, player i's utility function is continuous, we can assume without loss of generality that $\hat{m}_i^4 \in \Delta^*(A) = M_i^4$.

Since player i's utility is strictly higher under $h(m_i, m_{-i})$ than under $g(m_i, m_{-i})$, for every $(\theta, m_{-i}) \in$Supp$(\underset{\Theta \times M_{-i}}{marg} \pi_i(t_i))$, player i can change m_i with $m_i' \in M_i$, where its fourth component is $\hat{m}_i^4$ and its second component is $\hat{m}_i^2 > 1$, so that he can trigger *Rule 3*. Since the utility gain of player i is obtained point-wise in the Supp$(\underset{\Theta \times M_{-i}}{marg} \pi_i(t_i))$,

[33]When *Rule 3* applies, we can see, by the arguments provided above, that player i can find a profitable deviation.

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we obtain the desired contradiction. Thus, $\pi_i(t_i) \in \Delta_{t_i}^{\tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}} (\Theta \times T_{-i} \times M_{-i})$.

Finally, let us show that $m_i^{11} \in \beta_i^{\alpha+1}(t_i)$. Since $\pi_i(t_i) \in \Delta_{t_i}^{\tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}} (\Theta \times T_{-i} \times M_{-i})$, we have that

$$\sum_{t_{-i}\in T_{-i}} \sum_{m_{-i}\in \tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}(t_{-i})} \pi_i(t_i)\left[\theta, t_{-i}, m_{-i}\right] = 1.$$

Let $\nu_i(t_i) \in \Delta\left(\Theta \times T_{-i} \times \hat{T}\right)$ be defined by

$$\nu_i(t_i)\left[\theta, t_{-i}, m_{-i}^1\right] = \sum_{m_{-i}\in \tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}\left(m_{-i}^1\right)} \pi_i(t_i)\left[\theta, t_{-i}, m_{-i}\right]. \tag{71}$$

By definition, we can see that $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^{\alpha}}\left(\Theta \times T_{-i} \times M_{-i}^1\right)$. Since $m_1^2 = 1$, then *Rule 1* applies with probability 1, and so

$$\begin{aligned}\textstyle\sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\mathrm{marg}_{\Theta\times M_{-i}}\pi_i(t_i)\left[\theta, m_{-i}\right]\right) u_i\left(g\left(m_i, m_{-i}\right), \theta\right) &= \\ \textstyle\sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\mathrm{marg}_{\Theta\times M_{-i}}\pi_i(t_i)\left[\theta, m_{-i}\right]\right) u_i\left(f\left(m_i^1, m_{-i}^1\right), \theta\right),\end{aligned}$$

and so, by (71),

$$\begin{aligned}\textstyle\sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\mathrm{marg}_{\Theta\times M_{-i}}\pi_i(t_i)\left[\theta, m_{-i}\right]\right) u_i\left(f\left(m_i^1, m_{-i}^1\right), \theta\right) &= \\ \textstyle\sum_{\left(\theta,m_{-i}^1\right)\in\Theta\times M_{-i}^1} \left(\mathrm{marg}_{\Theta\times M_{-i}^1}\nu_i(t_i)\left[\theta, m_{-i}^1\right]\right) u_i\left(f\left(m_i^1, m_{-i}^1\right), \theta\right).\end{aligned}$$

Moreover, since $m_i \in BR_i\left(\mathrm{marg}_{\Theta\times M_{-i}}\pi_i(t_i)|\mathcal{M}\right)$ and since, moreover, player i can never induce *Rule 3*, it follows from the definition of g that

$$\begin{aligned}\textstyle\sum_{\left(\theta,m_{-i}^1\right)\in\Theta\times M_{-i}^1} \left(\mathrm{marg}_{\Theta\times M_{-i}^1}\nu_i(t_i)\left[\theta, m_{-i}^1\right]\right) u_i\left(f\left(m_i^1, m_{-i}^1\right), \theta\right) &\geq \\ \textstyle\sum_{\left(\theta,m_{-i}^1\right)\in\Theta\times M_{-i}^1} \left(\mathrm{marg}_{\Theta\times M_{-i}^1}\nu_i(t_i)\left[\theta, m_{-i}^1\right]\right) u_i\left(m_i^3\left(m_{-i}^1\right), \theta\right),\end{aligned} \tag{72}$$

for all $m_i^3 \in Y_i^*$. Since Y_i^* is a countable, dense subset of $\bar{Y}_i^f$ and since u_i is continuous, we have that the inequality in (72) holds for all $m_i^3 \in \bar{Y}_i^f$. Since $\nu_i(t_i) \in \Delta_{t_i}^{\beta_{-i}^{\alpha}}\left(\Theta \times T_{-i} \times M_{-i}^1\right)$ and since, moreover, the inequality in (72) holds for all $m_i^3 \in \bar{Y}_i^f$, and $m_i^{11} \in \beta_i^{\alpha}(t_i)$, it follows from (42) that $m_i^{11} \in \beta_i^{\alpha+1}(t_i)$, as we sought. □

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Lemma 27. For all $\alpha \in \Omega$ and all $i \in \mathcal{I}$, $S_i^{\alpha,\mathcal{M},\mathcal{T}} \subseteq \Sigma_i^{\beta_i^\alpha}$

Proof. Let us proceed by transfinite induction over Ω. It is clear that $S_i^{\alpha,\mathcal{M},\mathcal{T}} \subseteq \Sigma_i^{\beta_i^\alpha} = M_i$ for all $i \in \mathcal{I}$ if $\alpha = 0$. Fix any $\alpha \in \Omega \setminus \{0\}$. Suppose that for all $\gamma < \alpha$, $S_i^{\gamma,\mathcal{M},\mathcal{T}} \subseteq \Sigma_i^{\beta_i^\gamma}$ for all $i \in \mathcal{I}$. Fix any $i \in \mathcal{I}$. We proceed according to whether α is a successor ordinal or not.

Suppose that α is a limit ordinal. Since $\bigcap_{\gamma<\alpha} S_i^{\gamma,\mathcal{M},\mathcal{T}} = S_i^{\alpha,\mathcal{M},\mathcal{T}}$, by Definition 3, it follows that $S_i^{\alpha,\mathcal{M},\mathcal{T}} \subseteq \bigcap_{\gamma<\alpha} \Sigma_i^{\beta_i^\gamma}$. Fix any $t_i \in T_i$ and any $m_i \in \bigcap_{\gamma<\alpha} \Sigma_i^{\beta_i^\gamma}(t_i)$. Then, $m_i^{11} \in \bigcap_{\gamma<\alpha} \beta_i^\gamma(t_i)$. It follows from (43) that $m_i^{11} \in \beta_i^\alpha(t_i)$. Since the choice of $t_i \in T_i$ was arbitrary, we have that $\bigcap_{\gamma<\alpha} \Sigma_i^{\beta_i^\gamma} \subseteq \Sigma_i^{\beta_i^\alpha}$. Since $S_i^{\alpha,\mathcal{M},\mathcal{T}} \subseteq \bigcap_{\gamma<\alpha} \Sigma_i^{\beta_i^\gamma}$, we have that $S_i^{\alpha,\mathcal{M},\mathcal{T}} \subseteq \Sigma_i^{\beta_i^\alpha}$.

Suppose that α is a successor ordinal. Fix any $t_i \in T_i$. We proceed according to whether $t_i \in T_i(\beta^{\alpha-1})$ or not. Suppose that $t_i \in T_i(\beta^{\alpha-1})$. Fix any $m_i \in S_i^{\alpha,\mathcal{M},\mathcal{T}}(t_i)$. The inductive hypothesis implies that $S_{-i}^{\alpha-1,\mathcal{M},\mathcal{T}} \subseteq \Sigma_{-i}^{\beta_{-i}^{\alpha-1}}$.

Since $m_i \in S_i^{\alpha,\mathcal{M},\mathcal{T}}(t_i)$, Definition 3 implies that $m_i \in S_i^{\alpha-1,\mathcal{M},\mathcal{T}}$ and that there exists $\pi_i \in \Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i})$ such that $\pi_i(t_i) \in \Delta_{t_i}^{S_{-i}^{\alpha-1,\mathcal{M},\mathcal{T}}}(\Theta \times T_{-i} \times M_{-i})$ and that $m_i \in\mathrm{BR}_i\left(\mathrm{marg}_{\Theta\times M_{-i}}\pi_i(t_i)|\mathcal{M}\right)$. Since $S_{-i}^{\alpha-1,\mathcal{M},\mathcal{T}} \subseteq \Sigma_{-i}^{\beta^{\alpha-1}}$, it follows that

$$\pi_i(t_i) \in \Delta_{t_i}^{\Sigma_{-i}^{\beta^{\alpha-1}}}(\Theta \times T_{-i} \times M_{-i}).$$

Since $t_i \in T_i(\beta^{\alpha-1})$ and since, moreover, $m_i \in BR_i\left(\mathrm{marg}_{\Theta\times M_{-i}}\pi_i(t_i)|\mathcal{M}\right)$ and $\pi_i(t_i) \in \Delta_{t_i}^{\Sigma_{-i}^{\beta_{-i}^{\alpha-1}}}(\Theta \times T_{-i} \times M_{-i})$, Lemma 26 implies that $m_i^2 = 1$ and that $m_i^{11} \in \beta_i^\alpha(t_i)$. Thus, $m_i \in \Sigma_i^{\beta_i^\alpha}(t_i)$.

Suppose that $t_i \in T_i^c(\beta^{\alpha-1})$. Lemma 20 implies that $\beta_i^\alpha(t_i) = \bar{\beta}_i(t_i)$. It follows from (55) that $S_i^{\alpha,\mathcal{M},\mathcal{T}}(t_i) \subseteq \Sigma_i^{\beta^\alpha}(t_i)$.

Since the choice of player i and of player i's type t_i were arbitrary, we conclude that for all $i \in \mathcal{I}$, $S_i^{\alpha,\mathcal{M},\mathcal{T}} \subseteq \Sigma_i^{\beta_i^\alpha}$. By the principle of transfinite induction, the statement follows. □

Lemma 28. For all $\alpha \in \Omega$, all $i \in \mathcal{I}$, and all $t_i \in T_i(\beta^\alpha)$, if $m_i \in S_i^{\alpha+1,\mathcal{M},\mathcal{T}}(t_i)$, then $m_i^2 = 1$ and $m_i^{11} \in \beta_i^{\alpha+1}(t_i)$.

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Proof. Fix $(\alpha, i, t_i) \in \Omega \times \mathcal{I} \times T_i(\beta^\alpha)$. Suppose that $m_i \in S_i^{\alpha+1,\mathcal{M},\mathcal{T}}(t_i)$. Definition 3 implies that there exists $\pi_i(t_i) \in \Delta_{t_i}^{S_{-i}^{\alpha,\mathcal{M},\mathcal{T}}}(\Theta \times T_{-i} \times M_{-i})$ such that $m_i \in BR_i\left(\text{marg}_{\Theta\times M_{-i}}\pi_i(t_i)|\mathcal{M}\right)$. Lemma 27 implies that

$$S_{-i}^{\alpha,\mathcal{M},\mathcal{T}} \subseteq \Sigma_{-i}^{\beta_{-i}^{\alpha}}, \tag{73}$$

and so $\pi_i(t_i) \in \Delta_{t_i}^{\Sigma_{-i}^{\beta_{-i}^{\alpha}}}(\Theta \times T_{-i} \times M_{-i})$. Lemma 26 implies that $m_i^2 = 1$ and that $m_i^{11} \in \beta_i^{\alpha+1}(t_i)$.

Since the choice of $(\alpha, i, t_i) \in \Omega \times \mathcal{I} \times T_i(\beta^\alpha)$ was arbitrary, the proof is complete. □

Let us show that $\mathcal{M}$ ICR-implements f on $\mathcal{T}$. Lemma 25 implies that for all $i \in \mathcal{I}$ and $t_i \in T_i$, $S_i^{\mathcal{M},\mathcal{T}}(t_i) \neq \emptyset$. Thus, part (i) of Definition 4 is satisfied. Recall that Lemma 16 implies that there exists an α such that $\beta^\alpha = \beta^{\alpha+1} = \beta^*$. Recall that to avoid trivial cases, we are under the assumption that $T(\beta^*) = T$. Thus, $T(\beta^\alpha) = T$. Fix any $t \in T$ and any $m \in S^{\mathcal{M},\mathcal{T}}(t)$. Since $S^{\mathcal{M},\mathcal{T}}(t) \subseteq S^{\alpha+1,\mathcal{M},\mathcal{T}}(t)$, then $m \in S^{\alpha+1,\mathcal{M},\mathcal{T}}(t)$. Lemma 28 implies that $m_i^2 = 1$ and $m_i^{11} \in \beta_i^{\alpha+1}(t_i) = \beta_i^*(t_i)$ for all $(i, t_i) \in \mathcal{I} \times T_i$. *Rule 1* implies that $g(m) = f(m^1)$. Since f satisfies w-IIM on $\mathcal{T}$, it follows that β^* is an acceptable deception on $\mathcal{T}$ for f. This implies that $f(m^1) = f(t)$. Since the choice of $(t, m) \in T \times S^{\mathcal{M},\mathcal{T}}(t)$ was arbitrary, we conclude that part (ii) of Definition 4 is satisfied. Thus, f is ICR-implementable on $\mathcal{T}$.

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